Partial regularity of mass-minimizing rectifiable sections

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Abstract. Let B be a fiber bundle with compact fiber F over a compact Riemannian n-manifold M^n. There is a natural Riemannian metric on the total space B consistent with the metric on M. With respect to that metric, the volume of a rectifiable section $\sigma : M \to B$ is the mass of the image $\sigma(M)$ as a rectifiable n-current in B.

THEOREM 1. *For any homology class of sections of B, there is a mass-minimizing rectifiable current T representing that homology class which is the graph of a C^1 section on an open dense subset of M.*

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Introduction

The notion of the volume of a section of a fiber bundle over a manifold M was introduced by H. Gluck and W. Ziller, in the special case of the unit tangent bundle $\pi : T_1(M) \to M$, where sections are unit vector fields, or flows on M. The volume of a section σ is defined as the mass (Hausdorff n-dimensional measure) of the image $\sigma(M)$. They were able to establish, by constructing a calibration, that the tangents to the fibers of the standard Hopf fibration $S^3 \to S^2$ minimized volume among all sections of the unit tangent bundle of the round S^3.

However, in general calibrations are not available, even for the unit tangent bundles of higher-dimensional spheres. For a general bundle $\pi : B \to M$ over a Riemannian n-manifold M, with compact fiber F, there is a special class of rectifiable currents, called *rectifiable sections*, which includes all smooth sections and which has the proper compactness properties to guarantee the existence of volume-minimizing rectifiable sections in any homology class. Partial regularity of volume-minimizing rectifiable sections in general is the subject of this paper.

The basic partial-regularity result established here is that a volume-minimizing rectifiable section exists in any homology class of sections which is a C^1 section over an open, dense subset of M. This does not state that a dense subset of the section itself consists of regular points. In fact, there are simple counter-examples of that statement. Denseness of the set of points in M over which the section is regular is straightforward, but openness in M requires some work.

Our approach to this problem begins with a penalty functional, composed of the n-dimensional area integrand plus a parameter $(1/\epsilon)$ multiplied by a term measuring the deviation from a graph of a current in the total space. Each penalty functional will have energy-minimizing currents which are rectifiable currents in the total space, but which are not necessarily rectifiable sections. As the penalty parameter ϵ approaches 0, the “bad" set of points in the base over which the current is not a section will have small measure, and outside a slightly larger set the current will be a C^1 graph. These penalty minimizers will converge to a rectifiable section which will be a minimizer of the volume problem.

Once fundamental monotonicity properties are established for this limiting minimizer, the program to establish partial regularity of energy-minimizing currents due to Bombieri in [2] can be applied,

with significant modifications for the current situation, to show that the limiting minimizer is sufficiently smooth on an open dense set.

The main theorem of this paper is the following:

THEOREM 0.1. *Let B be a fiber bundle with compact fiber F over a compact Riemannian manifold M, endowed with the Sasaki metric from a connection on B. For any homology class of sections of B, there is a mass-minimizing rectifiable section T representing that homology class which is the graph of a C^1 section on an open dense subset of M.*

1. Definitions

Let B be a Riemannian fiber bundle with compact fiber F over a Riemannian n-manifold M, with projection $\pi : B \to M$ a Riemannian submersion. F is a j-dimensional compact Riemannian manifold. Following [10], B embeds isometrically in a vector bundle $\pi : E \to M$ of some rank $k \geq j$, which has a smooth inner product $< , >$ on the fibers, compatible with the Riemannian metric on F. The inner product defines a collection of connections, called *metric connections*, which are compatible with the metric. Let a metric connection ∇ be chosen. The connection ∇ defines a Riemannian metric on the total space E so that the projection $\pi : E \to M$ is a Riemannian submersion and so that the fibers are totally geodesic and isometric with the inner product space $E_x \cong \mathbb{R}^k$ [14], [6].

We will be using multiindices $\alpha = (\alpha_1, \ldots, \alpha_{n-l})$, $\alpha_i \in \{1, \ldots, n\}$ with $\alpha_1 < \cdots < \alpha_{n-l}$, over the local base variables, and $\beta = (\beta_1, \ldots, \beta_l)$, $\beta_j \in \{1, \ldots, k\}$ with $\beta_1 < \cdots < \beta_l$, over the local fiber variables (we will at times need to consider the vector bundle fiber, as well as the compact fiber F; which is considered will be clear by context). The range of pairs (α, β) is over all pairs satisfying $|\beta| + |\alpha| = n$, where $|(\alpha_1, \ldots, \alpha_m)| := m$. As a notational convenience, denote by n the n-tuple $n := (1, \ldots, n)$, and denote the null 0-tuple by 0.

In addition, we indicate by $A \ll B$ that A is bounded above by a constant times B, where that constant is independent of the variables included in A and B.

Definition 1.1. An n-dimensional current T on a Riemannian fiber bundle B over a Riemannian n-manifold M locally, over a coordinate neighborhood Ω on M, decomposes into a collection, called *components*, or *component currents of* T, with respect to the bundle structure. Given local coordinates (x, y) on $\pi^{-1}(\Omega) = \Omega \times \mathbb{R}^k$ and a smooth n-form $\omega \in E^n(\Omega \times \mathbb{R}^k)$, $\omega := \omega_{\alpha\beta} dx^\alpha \wedge dy^\beta$, define auxiliary currents $E_{\alpha\beta}$ by $E_{\alpha\beta}(\omega) := \int \omega_{\alpha\beta} d \|T\|$, where $\|T\|$ is the measure $\theta \mathcal{H}^n \llcorner Supp(T)$, with $\mathcal{H}^n$ Hausdorff n-dimensional measure in $\Omega \times \mathbb{R}^k$ and θ the multiplicity of T [11, pp 45-46]. The *component currents* of T are defined in terms of *component functions* $t_{\alpha\beta} : \Omega \times \mathbb{R}^k \to \mathbb{R}$ and the auxiliary currents, by:

$$T|_{\pi^{-1}(\Omega)} := \{T_{\alpha\beta}\} := \{t_{\alpha\beta} E_{\alpha\beta}\}.$$

The component functions $t_{\alpha\beta} : \pi^{-1}(\Omega) \to \mathbb{R}$ determine completely the current T, and the pairing between T and an n-form $\omega \in E^n(B) \llcorner \Omega \times \mathbb{R}^k$ is given by:

$$T(\omega) := \int_{\Omega \times \mathbb{R}^k} \sum_{\alpha\beta} t_{\alpha\beta} \omega_{\alpha\beta} d \|T\|.$$

Definition 1.2. A bounded current T in B is *a quasi-section* if, for each coordinate neighborhood $\Omega \subset M$,

1. $t_{n0} \geq 0$ for $\|T\|$-almost all points $p \in Supp(T)$, that is $< \overrightarrow{T}(q), \mathbf{e}(q) > \geq 0$, $\|T\|$-almost everywhere; where $\mathbf{e}(q) := \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n} \Big/ \left\| \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n} \right\|$ is the (unique) horizontal n-plane at q whose orientation is preserved under π_*, and $\overrightarrow{T}$ is the unit orienting n-vector field of T.
2. $\pi_\#(T) = 1[M]$ as an n-dimensional current on M.
3. $\partial T = 0$ (equivalently, for any $\Omega \subset M$, $\partial T \llcorner \pi^{-1}(\Omega)$ has support contained in $\partial \pi^{-1}(\Omega)$).

There is an $M > 0$ so that the fiber bundle B is contained in the disk bundle $E_M \subset E$ defined by $E_M := \{v \in E \,|\, \|v\| < M\}$, by compactness of B. Define the space $\widetilde{\Gamma}(E)$ to be the set of all countably rectifiable, integer multiplicity, n-dimensional currents which are quasi-sections in E, with support contained in E_M, called *(bounded) rectifiable sections* of E. The space $\Gamma(E)$ of *(strongly) rectifiable sections of* E is the smallest sequentially, weakly-closed space containing the graphs of C^1 sections of E which are supported within E_M.

A quasi-section which also is rectifiable is an element of $\widetilde{\Gamma}(E)$. It would seem to be a strictly stronger condition to be in $\Gamma(E)$, however, it is shown in [3] that, over a bounded domain Ω, $\Gamma(\Omega \times \mathbb{R}^k) = \widetilde{\Gamma}(\Omega \times \mathbb{R}^k)$. The norm defined in [3] is finite in this case since all currents have support contained in E_M. This extends to the statement that $\Gamma(E) = \widetilde{\Gamma}(E)$ for a vector bundle over a compact manifold M, since any such can be decomposed into finitely many bounded domains where the bundle structure is trivial, by a partition of unity argument.

The space $\widetilde{\Gamma}(B)$ of *rectifiable sections* of B is the subset of $\widetilde{\Gamma}(E)$ of currents with support in B, which is a weakly closed condition with respect to weak convergence. This follows since, for any point z outside of B, there is a smooth form supported in a neighborhood of z disjoint from B. The space $\Gamma(B)$ of *strongly* rectifiable sections is the smallest sequentially, weakly-closed space containing the graphs of C^1 sections of B. Since the fibers of B are compact, as is the base manifold M, minimal-mass elements will exist in $\widetilde{\Gamma}(B)$ or $\Gamma(B)$, and mass-minimizing sequences within any homology class will have convergent subsequences in $\widetilde{\Gamma}(B)$ or $\Gamma(B)$. This follows from lower semi-continuity with respect to convergence of currents, convexity of the mass functional, and the closure and compactness theorems for rectifiable currents. For compact manifolds, as above, $\widetilde{\Gamma}(E) = \Gamma(E)$, but it is *not* the case that $\widetilde{\Gamma}(B) = \Gamma(B)$ in general.

PROPOSITION 1.3. *Let $\{T_j\} \subset \widetilde{\Gamma}(B)$ (resp, $\Gamma(B)$) be a sequence with equibounded mass. Then, there is a subsequence which converges weakly to a current T in $\widetilde{\Gamma}(B)$ (resp, $\Gamma(B)$) .*

Proof. The Federer-Fleming compactness and closure theorems (see also [9, Theorem 7.61, pp. 204-5] shows that a weak subsequence limit will exist and will be a countably-rectifiable, integer-multiplicity current, with no interior boundaries. Since the map $\pi : B \to M$ is proper, $\pi_\#$ will then commute with weak limits, and so $\pi_\#(T) = 1[M]$. Similarly, the conditions $< \overrightarrow{T}(q), \mathbf{e}(q) > \geq 0$ $\|T\|$-almost everywhere and $Supp(T) \subset B$ are directly seen to be preserved under weak limits, so the limit will be in $\widetilde{\Gamma}(B)$. ☐

Definition 1.4. Given a current T, the induced measures $\|T\|$ and $\|T_{\alpha\beta}\|$ are defined locally by:

$$\|T_{\alpha\beta}\|(A) := \sup\left(T_{\alpha\beta}(\omega)\right), \text{ and}$$
$$\|T\|(A) := \sup\left(\sum_{\alpha\beta} T_{\alpha\beta}(\omega)\right),$$

where the supremum in either case is taken over all n-forms on B, $\omega \in E^n_0(B)$, with $comass(\omega) \leq 1$ [5, 4.1.7]and $Supp(\omega) \subset A$.

2. Coordinatizability

Let $T \in \widetilde{\Gamma}(B)$ have finite mass. Then, for each $x \in M$, we say that T is *coordinatizable over* x if there is an $r > 0$ so that $T \llcorner \pi^{-1}(B(x,r))$ (note that $\pi^{-1}(B(x,r)) \cong B(x,r) \times F$) has support contained within $B(x,r) \times U$, where $U \subset F$ is a contractible coordinate neighborhood of F, $U \cong \mathbb{R}^j$.

PROPOSITION 2.1. *The set of all points $x \in M$ where T is coordinatizable over x is an open, dense subset of M.*

Proof. Openness follows from the definition, which involves open neighborhoods in M. Note that the closed nested sets $Supp(T) \cap \pi^{-1}(\overline{B(x,r)})$, as $r \to 0$, have a nonempty intersection of $Supp(T) \cap \pi^{-1}(x)$. So, given any neighborhood U of $Supp(T) \cap \pi^{-1}(x)$ in F, for some $r > 0$ $\pi_2\left(Supp(T) \cap \pi^{-1}(B(x,r)\right) \subset U$, where π_2 is the projection of $\pi^{-1}(B(x,r_0)) \cong B(x,r_0) \times F$ onto F. Certainly if $Supp(T) \cap \pi^{-1}(x)$ is finite, then, since any finite set in F is contained in a contractible coordinate neighborhood in F, T will be coordinatizable at x. So, any point x over which T is not coordinatizable must have a preimage under π which is infinite, thus having infinite 0-dimensional Hausdorff measure. But, for

$$N := \{x \in M \,|\, T \,\text{is}\, \textit{not}\, \text{coordinatizable over x}\},$$

then, if N has positive Lebesgue measure on M, and if $\mathcal{F}$ is the volume (or mass) integrand,

$$\begin{aligned} \mathcal{F}(T) &:= \int_B d\,\|T\| \\ &\geq \int_{\pi^{-1}(N)} d\,\|T\| \\ &\geq \int_N \#(\pi^{-1}(x))dx \\ &= \infty, \end{aligned}$$

by the general area-coarea formula [11]. □

Remark 2.2. The Riemannian metric on $U \times V$ has the structure of a Riemannian submersion $\pi : U \times V \to U$, that is, the projection π is an isometry on the orthogonal complement to the fibers, and the projection onto the fiber, $\pi_2 : U \times V \to V$ is an isometry restricted to each fiber. The fiber metric is not necessarily Euclidean, and the orthogonal complements to the fibers will not necessarily form an integrable distribution, but that will not affect the arguments which follow.

3. Penalty Method

Let $\mathcal{F}(= \mathcal{M})$ be the standard volume (area) functional, applied to rectifiable sections. For an integer-multiplicity, countably-rectifiable current $T = \tau(M,\theta,\overrightarrow{T})$, where $M = Supp(T)$ and $\overrightarrow{T}$ is the unit orienting n-vector field, as in [11, p. 46]. Set, for each $\epsilon > 0$, the modified functional

$$\mathcal{F}_\epsilon(T) := \int_T f_\epsilon(\overrightarrow{T}) d\,\|T\|\,,$$

where $d\|T\| = \theta\mathcal{H}^n \llcorner Supp(T)$ and

$$f_\epsilon(\xi) := \|\xi\| + h_\epsilon(\xi) := \|\xi\| + \frac{1}{\epsilon}\left(|\xi_{n,0}| - \xi_{n,0}\right),$$

for $\xi \in \Lambda_n(T_*(B,z)) \cong \Lambda_n(\mathbb{R}^{n+k})$ ($\|\xi\|$ is the usual norm of ξ in $\Lambda_n(T_*(B,z))$ and $\xi_{n,0} :=< \xi, \mathbf{e} >$, where $\mathbf{e}$ is the unique unit horizontal n-plane so that $\pi_*(\mathbf{e}) = *dV_M$).

Note also that, since the original integrand is positive, so is f_ϵ, at any point ξ. Moreover, f_ϵ satisfies the homogeneity condition

$$f_\epsilon(t\xi) = tf_\epsilon(\xi)$$

for $t > 0$.

Set

$$\mathcal{H}_0(T) := \int_T h_0\left(\overrightarrow{T}\right) d\|T\|,$$

where $h_0(\xi) := (|\xi_{n,0}| - \xi_{n,0})$, and set

$$\mathcal{H}_\epsilon(T) := \frac{1}{\epsilon}\mathcal{H}_0(T).$$

On the parts of T which project to a negatively-oriented current (locally) on the base, the functional $\mathcal{H}_0()$ has value equal to twice the Lebesgue measure of the projected image, considered as measurable subsets of the base.

Clearly f_ϵ satisfies the bounds

$$\|\xi\| \leq f_\epsilon(\xi) \leq \left(1 + \frac{2}{\epsilon}\right)\|\xi\|.$$

In addition, the functional satisfies the λ-ellipticity condition with $\lambda = 1$

$$[\mathcal{M}(X) - \mathcal{M}(mD)] \leq \mathcal{F}_\epsilon(X) - \mathcal{F}_\epsilon(mD) \tag{1}$$

where mD is a flat disk with multiplicity m and X is a rectifiable current with the same boundary as mD. This inequality is clear if $\pi_\#(mD)$ is positively-oriented, since in that case $\mathcal{M}(mD) = \mathcal{F}_\epsilon(mD)$, and (in all cases) $\mathcal{M}(X) \leq \mathcal{F}_\epsilon(X)$. If $\pi_\#(mD)$ is negatively-oriented, though, then $\pi_\#(X) = \pi_\#(mD)$ since they have the same boundary and are integer-multiplicity countably-rectifiable n-currents on $\mathbb{R}^n$, by the constancy theorem. However, in this case $\mathcal{H}_\epsilon(X) \geq \mathcal{H}_\epsilon(mD)$, and the result follows.

3.1. Minimization problem

We now consider the mass-minimization problem for rectifiable sections $T \in \widetilde{\Gamma}(B)$ within a given integral homology class $[T] \in H_n(B, \mathbb{Z})$ which includes graphs, that is, for which there is a smooth section $S_0 \in \Gamma(B)$ with $[S_0] = [T]$. Set $A := \|S_0\|$. Set

$$\mathbf{R}[T] := \{S \in [T] | \, S \text{ is a countably rectifiable, integer-multiplicity } n\text{-current in } B\}.$$

For any $\epsilon > 0$, since the tangent planes at each point of S_0 projects to an n-plane of positive orientation, $h_\epsilon(\overrightarrow{S_0}) = \frac{1}{\epsilon}\left(|\xi_{n,0}| - \xi_{n,0}\right) = 0$, and so $\mathcal{F}_\epsilon(S_0) = \|S_0\| := A$, which shows that $\{S \in \mathbf{R}[T] \, |\mathcal{F}_\epsilon(S) \leq A\} \neq \emptyset$. Thus, if $B_0 := \{S \in \mathbf{R}[T] \, | \|S\| \leq 2A\}$,

$$Lev_A\mathcal{F}_\epsilon := \{S \in \mathbf{R}[T] \, |\mathcal{F}_\epsilon(S) \leq A\} \subset B_0,$$

since for any current $\mathcal{F}_\epsilon(S) \geq \|S\|$. Also, by the Federer-Fleming closure theorem, B_0 is compact with respect to the usual convergence of currents. Since the functional $\mathcal{F}_\epsilon$ is elliptic (eq (1)), it will be lower semi-continuous with respect to weak convergence of rectifiable currents [5, 5.1.5]. Thus each $Lev_A\mathcal{F}_\epsilon$ is compact in this topology, and so by [16], for each such ϵ, an $\mathcal{F}_\epsilon$-energy-minimizing rectifiable current $T_\epsilon \in [T]$ exists, and $\mathcal{F}_\epsilon(T_\epsilon) < \|S_0\| = A$.

Set

$$\begin{aligned} min(\mathcal{F}_\epsilon) &:= min\,\{\mathcal{F}_\epsilon(T)\,|T \in \mathbf{R}[T]\,\} \\ Argmin(\mathcal{F}_\epsilon) &:= \{T \in \mathbf{R}[T]\,|\mathcal{F}_\epsilon(T) = min(\mathcal{F}_\epsilon)\,\}\,, \\ min(\mathcal{F}) &:= min\,\{\,\mathcal{F}(T)|\,T \in [T] \cap \Gamma(B)\}\,, \text{ and} \\ Argmin(\mathcal{F}) &:= \{T \in [T] \cap \Gamma(B)\,|\mathcal{F}(T) = min(\mathcal{F})\,\}\,. \end{aligned}$$

Similarly to [13], we have

PROPOSITION 3.1. **[Convergence of the penalty problems]**

$$\begin{aligned} \lim_{\epsilon\downarrow 0} min\,(\mathcal{F}_\epsilon) &= min\,(\mathcal{F})\,, \\ \limsup_{\epsilon\downarrow 0} Argmin(\mathcal{F}_\epsilon) &\subset Argmin(\mathcal{F}). \end{aligned}$$

Remark 3.2. That is, the minimal values of the penalty functionals on that homology class converge to the minimum of the mass of all homologous rectifiable sections, and the limsup of the set of minimizing currents [13] of the penalty problems is contained in the set of mass-minimizing rectifiable sections. This does not imply that each mass-minimizing rectifiable section is the limit of a sequence of minimizers of the penalty problems, but that one such mass-minimizing rectifiable section is such a limit.

Proof. Since the set of countably-rectifiable integer-multiplicity currents in $[T]$ (the domain of $\mathcal{F}_\epsilon$) contains the rectifiable sections, and $\mathcal{F}_\epsilon(S) = \mathcal{F}(S) = \|S\|$ for any rectifiable section S, we have immediately that $min\,(\mathcal{F}_\epsilon) \leq min\,(\mathcal{F})$. Moreover, $min\,(\mathcal{F}_{\epsilon_1}) \leq min\,(\mathcal{F}_{\epsilon_2})$ if $\epsilon_1 > \epsilon_2$, since for T_{ϵ_i} minimizers of $\mathcal{F}_{\epsilon_i}$,

$$\mathcal{F}_{\epsilon_1}(T_{\epsilon_1}) \leq \mathcal{F}_{\epsilon_1}(T_{\epsilon_2}) \leq \mathcal{F}_{\epsilon_2}(T_{\epsilon_2}),$$

so $\lim_{\epsilon\downarrow 0} min(\mathcal{F}_\epsilon)$ exists.

Take some $T_\epsilon \in \mathbf{R}[T]$ which minimizes $\mathcal{F}_\epsilon$within $\mathbf{R}[T]$. Then

$$\begin{aligned} \|T_\epsilon\| &= \mathcal{F}_\epsilon(T_\epsilon) - H_\epsilon(T_\epsilon) \\ &\leq \mathcal{F}_\epsilon(T_\epsilon) \\ &= min(\mathcal{F}_\epsilon) \\ &\leq min(\mathcal{F}). \end{aligned}$$

This shows that $T_\epsilon \in B_0$ above, which, in the topology of weak convergence of countably-rectifiable, integer-multiplicity currents, is compact. So, by the Federer-Fleming compactness and closure theorems [5, 4.2.16, 4.2.17], some subsequence of $\{T_\epsilon\}$ converges as $\epsilon \downarrow 0$ to some $S \in B_0$.

Since the penalty component satisfies

$$\mathcal{H}_0(T_\epsilon) = \epsilon\mathcal{H}_\epsilon(T_\epsilon) = \epsilon\,(min(\mathcal{F}_\epsilon) - \|T_\epsilon\|)\,,$$

and the penalty functional $\mathcal{F}_\epsilon$ is lower semi-continuous with respect to weak convergence of currents, we have

$$\begin{aligned}\mathcal{H}_0(S) &\leq \liminf_{\epsilon\downarrow 0}(\mathcal{H}_0(T_\epsilon)) \\ &= \liminf_{\epsilon\downarrow 0} \epsilon\,(min(\mathcal{F}_\epsilon) - \|T_\epsilon\|) \\ &\leq \liminf_{\epsilon\downarrow 0} \epsilon\,(min(\mathcal{F})) \\ &= 0.\end{aligned}$$

So. immediately we have that $S \in \widetilde{\Gamma}(B)$, so that $\mathcal{F}(S) \geq min(\mathcal{F})$. Applying the same limit to the previous equation,

$$\begin{aligned}\mathcal{F}(S) &= \|S\| \\ &\leq \liminf_{\epsilon\downarrow 0} \|T_\epsilon\| \\ &= \liminf_{\epsilon\downarrow 0} (\mathcal{F}_\epsilon(T_\epsilon) - H_\epsilon(T_\epsilon)) \\ &\leq \liminf_{\epsilon\downarrow 0} \mathcal{F}_\epsilon(T_\epsilon) \\ &= \liminf_{\epsilon\downarrow 0} min(\mathcal{F}_\epsilon) \\ &\leq min(\mathcal{F}),\end{aligned}$$

which implies that all inequalities must be equalities, and S is a mass-minimizing rectifiable section in $[T] \cap \widetilde{\Gamma}(B)$. In addition, we get that

$$\lim_{\epsilon\downarrow 0} min(\mathcal{F}_\epsilon) = min(\mathcal{F})$$

and, any limit current of a subsequence of minimizers $\{T_\epsilon\}$(for a sequence of ϵ's going to 0) will be a minimizer T_0 of $\mathcal{F}$ *on* $[T] \cap \widetilde{\Gamma}(B)$. □

The set of points $B_\epsilon \subset \Omega$ where T_ϵ is not a section,

$$B_\epsilon := \pi\left(\left\{p \in Supp(T_\epsilon) |\, h_\epsilon(\overrightarrow{T}_p) > 0\right\}\right),$$

satisfies, where $\mathbf{e}$ is the horizontal n-plane,

$$\begin{aligned}\mathcal{H}_0(T_\epsilon) &= \epsilon\mathcal{H}_\epsilon(T_\epsilon) \\ &= \int_{\Omega\times F} h_0(T_\epsilon) d\,\|T_\epsilon\| \\ &= \int_{\Omega\times F} (\left|< \overrightarrow{T_\epsilon}, \mathbf{e} >\right| - < \overrightarrow{T_\epsilon}, \mathbf{e} >) d\,\|T_\epsilon\| \\ &= -2\int_{B_\epsilon\times F} < \overrightarrow{T_\epsilon}, \mathbf{e} > d\,\|T_\epsilon\| \\ &= 2\int_{B_\epsilon\times F} \left|< \overrightarrow{T_\epsilon}, \mathbf{e} >\right| d\,\|T_\epsilon\| \\ &= 2T_\epsilon\left(\pi^*(d\,V_\Omega|_{B_\epsilon})\right) \\ &= 2\pi_\#(T_\epsilon)(d\,V_\Omega|_{B_\epsilon}) \\ &\geq 2\,\|B_\epsilon\|\,,\end{aligned} \tag{2}$$

where dV_Ω is the volume element of the base, since $\pi_\#(T_\epsilon)|_{B_\epsilon}$ is a (positive integer) multiple of the fundamental class of the base, restricted to B_ϵ. From the previous result,

$$\lim_{\epsilon\downarrow 0}\mathcal{H}_\epsilon(T_\epsilon)=0,$$

thus $\|B_\epsilon\|$ approaches 0 more rapidly than ϵ itself. Similarly to [16], we have the following:

LEMMA 3.3. $\|B_\epsilon\| \leq \frac{\epsilon}{|\ln(\epsilon)|}A$, *where* A *depends only on dimension and the homology class* $[T] \in H_n(B,\mathbb{Z})$.

Proof. If $v_\epsilon := \mathcal{F}_\epsilon(T_\epsilon)$, for $0<\epsilon_1<1$, then since v_ϵ is a monotone-decreasing function of ϵ, it is differentiable almost-everywhere, and

$$\begin{aligned}
|v'_\epsilon| &= \left|\lim_{h\to 0}\frac{v_\epsilon - v_{\epsilon-h}}{h}\right| \\
&\geq \left|\lim_{h\to 0}\frac{\mathcal{F}_\epsilon(T_\epsilon)-\mathcal{F}_{\epsilon-h}(T_\epsilon)}{h}\right| \\
&= \left|\lim_{h\to 0}\frac{\left(\frac{1}{\epsilon}-\frac{1}{\epsilon-h}\right)}{h}\right|\mathcal{H}_0(T_\epsilon) \\
&= \frac{1}{\epsilon^2}\mathcal{H}_0(T_\epsilon).
\end{aligned}$$

In addition, for any fixed rectifiable section S in the homology class $[T]$, for all $\epsilon>0$ $\nu_\epsilon \leq \mathcal{F}(S)$, so that ν_ϵ is bounded.

Now, as in [16, p. 70, Theorem 7.3],

$$\begin{aligned}
C &\geq v_{\epsilon_1} - v_1 \\
&\geq \int_{\epsilon_1}^{1} |v'_\epsilon|\, d\epsilon \\
&\geq \int_{\epsilon_1}^{1} \text{ess inf}_{\epsilon_1<\epsilon<1}\left(\epsilon\,|v'_\epsilon|\right)\frac{1}{\epsilon}d\epsilon \\
&= \text{ess inf}_{\epsilon_1<\epsilon<1}\epsilon\,|v'_\epsilon|\cdot(-\ln(\epsilon_1)) \\
&\geq \text{ess inf}_{\epsilon_1<\epsilon<1}\frac{1}{\epsilon}\mathcal{H}_0(T_\epsilon)\cdot|\ln(\epsilon_1)| \\
&\geq \text{ess inf}_{\epsilon_1<\epsilon<1}\frac{1}{\epsilon}\,\|B_\epsilon\|\cdot|\ln(\epsilon_1)|\,.,
\end{aligned}$$

applying (2). Since $v_{\epsilon_1}-v_1$is bounded (and nonnegative), there is a constant C so that

$$\|B_\epsilon\| \leq \frac{\epsilon}{|\log(\epsilon)|}C,$$

where C depends only on the homology class $[T]$ of sections being considered. □

4. Existence of tangent cones

Let T be a mass-minimizing rectifiable section, and presume that T is the limit of a sequence T_{ϵ_i} of minimizers of the penalty energy $\mathcal{F}_{\epsilon_i}$. (At least one minimizer of the mass functional among rectifiable sections is of this form), by Proposition (3.1).

PROPOSITION 4.1. *For any point $p \in Supp(T)$, the mass-density $\Theta(p,T)$ is at least 1. Moreover, there is a (possibly non-unique) tangent cone at p of T.*

Remark 4.2. The proof will depend on a monotonicity of mass ratio result. Once that is established, the result will follow similarly to the case for area-minimizing rectifiable currents.

LEMMA 4.3. **[Monotonicity of mass ratio].** *For any $p \in Supp(T)$, the ratio*

$$\frac{\mathcal{F}\left(T \llcorner B(p,r)\right)}{r^n}$$

is a monotone increasing function of r.

Proof. (of the Lemma). Consider, for a sequence $\epsilon = \epsilon_i$ converging to 0, the penalty energy function

$$f_\epsilon(r) := \mathcal{F}_\epsilon\left(T_\epsilon \llcorner B(p_\epsilon, r)\right),$$

where $p_\epsilon \in Supp(T_\epsilon)$. We show that the penalty function satisfies the monotonicity differential inequality $(f_\epsilon(r)/r^n)' \geq 0$, as in [11].

Choose a radius r for which the boundary $\partial\left(T_\epsilon \llcorner B(p_\epsilon, r)\right)$ is rectifiable (true for almost-all r by slicing). For such an r, note that $\partial(T_\epsilon \llcorner B(p_\epsilon, r))$ is the boundary of the restriction of T_ϵ to the ball. Let $C[\partial(T_\epsilon \llcorner B(p_\epsilon, r))]$ be the cone over $\partial(T_\epsilon \llcorner B(p_\epsilon, r))$ with cone point p_ϵ, oriented so that $C[\partial(T_\epsilon \llcorner B(p_\epsilon, r))] + T_\epsilon \llcorner (B \backslash B(p_\epsilon, r))$ is a cycle. Define a boundary penalty-energy $\partial\mathcal{F}_\epsilon$ by restriction, that is:

$$\partial\mathcal{F}_\epsilon(\partial(T_\epsilon \llcorner B(p_\epsilon, r))) := \int_B \left\|\overrightarrow{T_\epsilon}\right\| + \frac{1}{\epsilon}\left(\left|< \overrightarrow{T_\epsilon}, \mathbf{e} >\right| - < \overrightarrow{T_\epsilon}, \mathbf{e} >\right) d\left\|\partial(T_\epsilon \llcorner B(p_\epsilon, r))\right\|.$$

Since $C_r := C[\partial((T_\epsilon \llcorner G_\epsilon) \llcorner B(p_\epsilon, r))]$ is a cone,

$$\begin{aligned} \mathcal{F}_\epsilon(C_r) &\leq \frac{n}{r}\partial\mathcal{F}_\epsilon(\partial C_r) \\ &= \frac{n}{r}\partial\mathcal{F}_\epsilon(\partial(T_\epsilon \llcorner B(p_\epsilon, r))). \end{aligned}$$

Now, set

$$f_\epsilon(r) := \mathcal{F}_\epsilon(T_\epsilon \llcorner B(p_\epsilon, r)).$$

We claim that slicing by $u(x) = \|x - p_\epsilon\|$ yields that, for almost-every r (as above)

$$\partial\mathcal{F}_\epsilon(\partial(T_\epsilon \llcorner B(p_\epsilon, r))) \leq f'_\epsilon(r).$$

To show this, let T be a rectifiable current, and u Lipschitz. The slice

$$< T, u, r+ > := \partial T \llcorner \{x \,|u(x) > r\} - \partial(T \llcorner \{x \,|u(x) > r\})$$

satisfies, for $\partial\mathcal{H}_0(< T, u, r+ >) := \int_B \left(\left|< \overrightarrow{T}, \mathbf{e} >\right| - < \overrightarrow{T}, \mathbf{e} >\right) d\,\|< T, u, r+ >\|$, the following:

$$\begin{aligned} \partial\mathcal{H}_0(< T, u, r+ >) &\leq Lip(u) \liminf_{h \downarrow 0} \mathcal{H}_0(T) \llcorner \{r < u(x) < r + h\}/h \\ &= Lip(u)\frac{\partial}{\partial r}\mathcal{H}_0(T) \llcorner \{x \,|u(x) \leq r\}, \end{aligned}$$

where we have abused notation and denoted the Dini derivative in the previous line by $\partial/\partial r$. This follows by considering, for a small, positive h, a smooth approximation f of the characteristic function of $\{x\,|u(x) > r\}$ with

$$f(x) = \begin{cases} 0, & \text{if } u(x) \leq r \\ 1, & \text{if } u(x) \geq r+h \end{cases}$$

and $Lip(f) \leq Lip(u)/h$. Then (cf. [11, 4.11, p. 56])

$$\begin{aligned}
\partial\mathcal{H}_0(< T, u, r+ >) &\approx \partial\mathcal{H}_0((\partial T)\llcorner f - \partial(T\llcorner f)) \\
&= \partial\mathcal{H}_0(T\llcorner df) \\
&\leq Lip(f)\mathcal{H}_0(T)\llcorner\{r < u(x) < r+h\} \\
&\lesssim Lip(u)\mathcal{H}_0(T)\llcorner\{r < u(x) < r+h\}/h \\
&= Lip(u)\frac{\partial}{\partial u}\mathcal{H}_0(T)\llcorner\,\{x\,|u(x) \leq r\}\,.
\end{aligned}$$

In the present case, with $u(x) := \|x - p_\epsilon\|$, $< T_\epsilon, u, r+ >= \partial(T_\epsilon\llcorner B(p_\epsilon, r))$, $\partial\mathcal{F}_\epsilon(\partial(T_\epsilon\llcorner B(p_\epsilon, r))) \leq f'_\epsilon(r)$ as claimed for almost-every r, since for the standard mass functional this result is standard, and $\mathcal{F}_\epsilon = \mathcal{M} + \frac{1}{\epsilon}\mathcal{H}_0$.

Combining these two relationships together and using minimality of T_ϵ,

$$f_\epsilon(r) := \mathcal{F}_\epsilon(T_\epsilon\llcorner B(p_\epsilon, r)) \leq \mathcal{F}_\epsilon(C[\partial(T_\epsilon\llcorner B(p_\epsilon, r))]) \leq \frac{n}{r}\partial\mathcal{F}_\epsilon(\partial(T_\epsilon\llcorner B(p_\epsilon, r))) \leq \frac{n}{r}\frac{df_\epsilon(r)}{dr},$$

for almost-every r, hence the absolutely continuous part of $f_\epsilon(r)/r^n$ is increasing. Since any singular part is due to increases in $f_\epsilon(r)$, $f_\epsilon(r)/r^n$ is increasing as claimed.

Let $p_\epsilon \to p$ be a sequence of points on the support of the penalty minimizers converging to $p \in Supp(T)$. Set $f(r) := \mathcal{F}\left(T\llcorner B(p, r)\right)$. Since $f_\epsilon(r)/r^n$ is monotone increasing as a function of r for each fixed $\epsilon > 0$, so will be $f(r)/r^n$. □

Arguing precisely as in [5, 5.4.3], (see also [11, pp. 90-95]), Proposition (4.1) follows.

5. Domain of the penalty-minimizers

Let $\Omega = B(x_0, R)$ be a ball. It follows from the structure theorem for rectifiable currents [5] that, except over the bad set B_ϵ, which is a set of mass less than ϵR^n, the penalty-minimizer T_ϵ will be the graph of a vector-valued BV function u_ϵ.

The points $x \in \Omega\backslash B_\epsilon$ so that for all $p = \pi^{-1}(x) \cap Supp(T_\epsilon)$, $\Theta(p) = 1$, has measure approaching that of Ω as $\epsilon \downarrow 0$ because of our bounds on B_ϵ. Since $T_\epsilon\llcorner\pi^{-1}(\Omega\backslash B_\epsilon)$ is a rectifiable section, the structure theorem for rectifiable currents implies that for Ω-a.e. points x of $\Omega\backslash B_\epsilon$, there is one point in $\pi^{-1}(x) \cap Supp(T_\epsilon)$.

Define u_ϵ as a vector-valued BV-function over $\Omega\backslash B_\epsilon$ whose carrier is $Supp(T_\epsilon)$ [2, Section IV], defined coordinatewise by integration, first defining S_j as n-dimensional currents in U by $S_j(\phi) := T(y_j\pi^*(\phi))$ for $\phi \in E^n(U)$ and y_j the j^{th} coordinate of the fiber (U must be a coordinatizable neighborhood). Then, the components of u_ϵ can be defined by $S_j(\phi) = \int(u_\epsilon)_j(x)\phi$, which define the components as BV_{loc}-functions on U.

It is not clear (compare [2, p. 106]) that this BV map will be a Lipschitz graph a.e. in general. For example, if T is the simple staircase current $T_\alpha = [[(t, \alpha \lfloor t \rfloor)]] + [[(\lfloor t \rfloor, \alpha(t-1))]]$, $t \in [0, n]$, $T \in \Gamma([0,n] \times \mathbb{R})$, then T will be a polyhedral chain, and so the image of a Lipschitz map. However, the set A on which $T \llcorner \pi^{-1}(A)$ will have a single point in each preimage is the base interval minus finitely many points (excluding the points that are the projections of the risers of the stairs), and $Supp(T) \cap \pi^{-1}(A)$ cannot be a Lipschitz graph on all of A. By controlling the height α of the risers the total cylindrical excess E of this example can be as small as needed as well.

However, it is the case that there will be, for any positive number $\delta > 0$, a Lipschitz map g so that $g = u_\epsilon$ except on a set of measure less than δ, by Theorem 2 page 252 of [4]. In fact, g can be taken to be C^1 by Corollary 1, p. 254, of the same reference. The Lipschitz constant of the map g will clearly depend upon δ, as is illustrated by the example above.

Now, it is not necessarily true that the graph of g will agree with $Supp(T_\epsilon)$ on the set where g agrees with u_ϵ, since that graph does not necessarily agree with $Supp(T_\epsilon)$ itself.

PROPOSITION 5.1. *For any $\epsilon > 0$, there is a set $D_\epsilon \supseteq B_\epsilon$ of measure less than $2\|B_\epsilon\|$ and a C^1 map $g_\epsilon : U \backslash D_\epsilon \to F$ so that, as rectifiable currents,*

$$graph(g_\epsilon) \llcorner \pi^{-1}(U \backslash D_\epsilon) = T_\epsilon \llcorner \pi^{-1}(U \backslash D_\epsilon).$$

Proof. For $\delta > 0$ sufficiently small, Choose g_ϵ by [4, Corollary 1, p. 254] to agree with u_ϵ on U except for a set of measure δ, and to be C^1 and Lipschitz there. Take D_ϵ to be the union of this set with B_ϵ, which if δ is chosen small enough will have measure bounded by $2\|B_\epsilon\|$. It suffices to show that these currents agree except over a set of measure 0 in the domain, outside of D_ϵ. However, if they disagree on a set A, within $U \backslash D_\epsilon$, of positive measure, then for some i, $g_j(x) = (u_\epsilon)_j(x)$ is different from $y_j(Supp(T_\epsilon) \cap \pi^{-1}(x))$ on A. However, for any form ϕ on the base, over any subset $V \subseteq U \backslash D_\epsilon$, since $\pi_\#(T) = 1 \cdot [U]$,

$$\begin{aligned}
& \int_V y_j(Supp(T_\epsilon) \cap \pi^{-1}(x))\, \phi \\
= & \int_{Supp(T_\epsilon) \cap \pi^{-1}(V)} y_j(Supp(T_\epsilon) \cap \pi^{-1}(x)) < \pi^*(\phi), \overrightarrow{T}_\epsilon > d\mathcal{H}^n \\
= & \int_{Supp(T_\epsilon) \cap \pi^{-1}(V)} < y_j \pi^*(\phi), T_\epsilon > d\mathcal{H}^n \\
= & \left(T_\epsilon \llcorner \pi^{-1}(V)\right)(y_j \pi^*(\phi)) \\
= & S_j(\phi) \\
= & \int_V (u_\epsilon)_j(x)\phi.
\end{aligned}$$

Since this equality must hold for all ϕ and $V \subset U \backslash D_\epsilon$ as above, the two functions must agree on a set of full measure. □

Note 5.2. The mass $\|D_\epsilon\|$ will satisfy

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \|D_\epsilon\| = 0$$

by the construction of both B_ϵ and the extension D_ϵ as defined in the proof of the previous result. Similarly, Lemma [3.3] will imply that

$$\|D_\epsilon\| \leq \frac{2\epsilon}{|\log(\epsilon)|} A,$$

with A depending only on dimension.

6. Homotopies and deformations

Let T^t be a one-parameter family of countably-rectifiable integer-multiplicity currents with $T^0 = T_\epsilon$, smooth in t. The derivative $h := \frac{d}{dt}\Big|_0 T^t$ at $t = 0$ is a current, but in general will not be a rectifiable current. The support of h will be T_ϵ, but h will be represented by integration as

$$h(\phi) := \int_E < \overrightarrow{h}, \phi > d\,\|T_\epsilon\|\,,$$

where

$$\overrightarrow{h}\,d\,\|T_\epsilon\| = \frac{d}{dt}\Big|_0 \overrightarrow{T^t} d \left\|T^t\right\|.$$

If T_ϵ is a smooth graph, $T_\epsilon = graph(g_\epsilon)$, then T^t will be also, for t sufficiently small, $T^t = graph(g_\epsilon + tk + \mathcal{O}(t^2))$, by the implicit function theorem, and

$$\begin{aligned}
\overrightarrow{h}\,d\,\|T_\epsilon\| &= \frac{d}{dt}\Big|_0 \overrightarrow{T^t} d \left\|T^t\right\| \\
&= \frac{d}{dt}\Big|_0 (e_1 + \nabla_1 g_\epsilon + t\nabla_1 k + t^2 *) \wedge \cdots \wedge (e_n + \nabla_n g_\epsilon + t\nabla_n k + t^2 *) d\mathcal{L}^n \\
&= (\nabla_1 k \wedge (e_2 + \nabla_2 g_\epsilon) \wedge \cdots \wedge (e_n + \nabla_n g_\epsilon) + \cdots + (e_1 + \nabla_1 g_\epsilon) \wedge \cdots \wedge \nabla_n k)\, d\mathcal{L}^n.
\end{aligned}$$

Remark 6.1. Note that this derivative is first-order with respect to the derivative Dk. The derivative will be first-order with respect to Dk for places where T_ϵ is not a graph, since, being rectifiable, Dk is a sum of terms of that sort.

Equivalently, we can consider maps $H_t : [0,1] \times U \times \mathbb{R}^j \to U \times \mathbb{R}^j$, ambient homotopies of the region into itself, and the push-forward $(H_t)_\#(T) = T^t$. Of particular interest will be in families which are *vertical* in the sense that $H_t(x,y) = (x, y + \eta(t,x))$ for some $\eta : [0,1] \times U \to \mathbb{R}^j$. These are, of course, in the graph case equivalent to families $T^t = graph(g_\epsilon + \eta(t,x))$.

6.1. Euler-Lagrange equations for T_ϵ

Restrict the deformations T^t to be, for each $\epsilon > 0$, deformations in the vertical directions only. For a rectifiable section, such a deformation will remain a section. If the domain $U = B(x_0, R)$, is a coordinatizable neighborhood, so that the fiber can be considered to be a compact subset of $\mathbb{R}^j$, and if coordinates are chosen so that $(x_0, \overline{y})$ is $(0,0)$ (for a particular value of $\overline{y}$ to be determined), then, following [2], a deformation given by $T^t = (H_{t,R})_\#(T_\epsilon)$, where

$$H_{t,R}(x,y) = (x, y + t\eta(x/R)), \tag{3}$$

so that, over $B(x_0,R)\backslash D_\epsilon$, $T_\epsilon \llcorner C(x_0,R) = graph(g_t)$, where $g_t(x) = g_\epsilon(x) + t\eta(x/R)$, and where $\eta : B(0,1) \to \mathbb{R}^k$ is a smooth test function with support within the open ball and with $\|\nabla\eta\| \le 1$ pointwise. Set $H_t := H_{t,1}$.

Over a set of full measure in $Supp(T_\epsilon)$ the tangent cone at $(x, g_\epsilon(x)) \in Supp(T_\epsilon)$ is an n-plane and is defined as usual from the graph of g_ϵ. Since the area functional, as a functional over the base, is then

$$\int_{\Omega\backslash D_\epsilon} \sqrt{1 + \|\nabla g_\epsilon\|^2 + \|\nabla g_\epsilon \wedge \nabla g_\epsilon\|^2 + \cdots + \left\| \nabla g_\epsilon \underbrace{\wedge \cdots \wedge}_{n} \nabla g_\epsilon \right\|^2}\, d\mathcal{L}^n,$$

then the Euler-Lagrange equations, obtained from calculus of variations methods (using a vertical variation $g_t(x) = g_\epsilon(x) + t\eta(x/R)$), is

$$
\begin{aligned}
&\frac{d}{dt}\Big|_0 \int_{\Omega\setminus D_\epsilon} \sqrt{1 + \|\nabla g_t\|^2 + \|\nabla g_t \wedge \nabla g_t\|^2 + \cdots + \left\|\nabla g_t \overbrace{\wedge\cdots\wedge}^{n} \nabla g_t\right\|^2}\, d\mathcal{L}^n \\
= &\int_{\Omega\setminus D_\epsilon} \left(\sum_i < \nabla_i g_\epsilon, \nabla_i \eta(x/R) > + \sum_{k=2}^{n} \sum_{i_1<\cdots<i_k,l,m} (-1)^{i_l+i_m} \left\langle \nabla_{i_1} g_\epsilon \wedge \overset{i_l}{\underset{\wedge}{\cdots}} \wedge \nabla_{i_k} g_\epsilon, \right.\right. \\
&\left.\left., \nabla_{i_1} g_\epsilon \wedge \overset{i_m}{\underset{\wedge}{\cdots}} \wedge \nabla_{i_2} g_\epsilon > \right\rangle < \nabla_{i_l} g_\epsilon, \nabla_{i_m} \eta(x/R) > \right) \Big/ \\
&\left(\sqrt{1 + \|\nabla g_\epsilon\|^2 + \|\nabla g_\epsilon \wedge \nabla g_\epsilon\|^2 + \cdots + \left\|\nabla g_\epsilon \overbrace{\wedge\cdots\wedge}^{n} \nabla g_\epsilon\right\|^2} \right) d\mathcal{L}^n
\end{aligned}
$$

Since the quadratic form A, at a fixed point x, defined by $(v, w) \mapsto < \nabla_v g_\epsilon, \nabla_w g_\epsilon > := < Av, w >$ is symmetric, there is an orthonormal basis $\{e_i\}$ of $T_*(U, x)$ for which $A_i := \nabla_{e_i} g_\epsilon = Ae_i$ are mutually orthogonal, simplifying the calculations above somewhat.

$$
\begin{aligned}
0 \;=\; &\int_{U\setminus D_\epsilon} \left(\sum_i < A_i, \nabla_i \eta(x/R) > + \right. \\
&\left. + \sum_{k=2}^{n} \sum_{i_1<\cdots<i_k,l=1\ldots k} \|A_{i_1}\|^2 \overset{i_l}{\underset{\wedge}{\cdots}} \|A_{i_k}\|^2 < A_{i_l}, \nabla_{i_l} \eta((x-x_0)/R) > \right) \Big/ \\
&\left(\sqrt{1 + \sum_{k=1}^{n} \sum_{i_1<\cdots<i_k} \|A_{i_1}\|^2 \cdots \|A_{i_k}\|^2} \right) d\mathcal{L}^n
\end{aligned}
$$

Additionally, since

$$
1 + \sum_{k=1}^{n} \sum_{i_1<\cdots<i_k} \|A_{i_1}\|^2 \cdots \|A_{i_k}\|^2 = \prod_{i=1}^{n} \left(1 + \|A_i\|^2\right),
$$

and similarly, for each j

$$
1 + \sum_{k=1}^{n-1} \sum_{i_1<\cdots<i_k, i_l\neq j} \|A_{i_1}\|^2 \cdots \|A_{i_k}\|^2 = \prod_{i=1, i\neq j}^{n} \left(1 + \|A_i\|^2\right),
$$

as a functional over the base,

$$
0 \;=\; \int_{U\setminus D_\epsilon} \sum_{j=1}^{n} \frac{< A_j, \nabla_j \eta(x/R) >}{1 + \|A_j\|^2} \left(\sqrt{1 + \sum_{k=1}^{n} \sum_{i_1<\cdots<i_k} \|A_{i_1}\|^2 \cdots \|A_{i_k}\|^2} \right) d\mathcal{L}^n
$$

As a parametric integrand, the Euler-Lagrange equations simplify, in this basis at each point, to

$$
\begin{aligned}
0 \;&=\; \frac{d}{dt}\Big|_0 \int_{\pi^{-1}(U\setminus D_\epsilon)} f_\epsilon(\overrightarrow{T^t})\, d\left\|T^t\right\| \\
&=\; \int_{\pi^{-1}(U\setminus D_\epsilon)} \sum_i \frac{< \nabla_i g_\epsilon, \nabla_i \eta(x/R) >}{1 + \|\nabla_i g_\epsilon\|^2} d\left\|T_\epsilon\right\|,
\end{aligned}
$$

where g_ϵ is the BV-carrier of T_ϵ on the good set. Note that, although not explicitly manifest, the derivative of $d\left\|T^t\right\|$ with respect to t is included in the formula above, since the preceding calculations are nonparametric on the good set.

On the bad set, the deformation is

$$\frac{d}{dt}\bigg|_0 \int_{\pi^{-1}(D_\epsilon)} f_\epsilon(\overrightarrow{T^t})d\left\|T^t\right\| = \int_{\pi^{-1}(D_\epsilon)} < \overrightarrow{T_\epsilon}, \overrightarrow{h_{t,R}} > d\left\|T_\epsilon\right\| + B,$$

where, since the deformation is vertical, $B = 0$. This follows since a vertical deformation, that is, $T^t := (H_t)_\#(T)$ for $H_t(x, y) = (x, y + t\eta(x/R))$, the boundary of the set where the penalty energy is nonzero, and the penalty-energy $\mathcal{H}_\epsilon$ itself, will not change under such a deformation. Also, the mass of that part of T_ϵ which is vertical (for which $\pi_\#(\overrightarrow{T_\epsilon}) = 0$) will also remain unchanged under such a deformation.

7. Squash-deformation

Let E be the cylindrical excess of the penalty-minimizer T_ϵ,

$$E := Exc(T_\epsilon; R, x_0) := \frac{1}{R^n}\left(\mathcal{M}(T\llcorner\pi^{-1}(B(x_0, R))) - \mathcal{M}(\pi_\#(T\llcorner\pi^{-1}(B(x_0, R))))\right),$$

and for a given R, $0 < R < 1$, define the non-homothetic dilation $\phi_R(x, y) = (\frac{x}{R}, \frac{y}{\sqrt{E}R}) = (X, Y)$ of the cylinder $\pi^{-1}(B(x_0, R))$ (we restrict to a coordinatizable neighborhood, so that the fiber can be considered to be a compact set within $\mathbb{R}^j$, and we assume without loss of generality that $x_0 = 0$), and set $T_{\epsilon,R} := (\phi_R)_\#\left(T_\epsilon\llcorner\pi^{-1}(B(x_0, R))\right)$. $T_{\epsilon,R}$ minimizes the penalty functional $\mathcal{F}_{\epsilon,R}$ defined by

$$\mathcal{F}_{\epsilon,R}(S) := \int_{\pi^{-1}(B(x,R))} E^{-1}R^{-n} f_\epsilon\left(\overrightarrow{\left(\phi_R^{-1}\right)_\#(S)}\right) d\left\|\left(\phi_R^{-1}\right)_\# S\right\|, \tag{4}$$

which contracts the current S back to the cylinder of radius R, evaluates the original penalty functional there, and scales to compensate for the factors of R and some of the factors of E. Consider the Euler-Lagrange equations of this functional, on $\widetilde{\Gamma}(B(x_0, 1) \times \mathbb{R}^k)$. Applying a vertical deformation as before,

$$\begin{aligned}
&\frac{d}{dt}\mathcal{F}_{\epsilon R}\left((h_t)_\#(T_{\epsilon,R})\right)\\
&= \frac{d}{dt}\int_{\pi^{-1}(B(x_0,R))} E^{-1}R^{-n} f_\epsilon\left(\overrightarrow{\left(\phi_R^{-1}\right)_\#\left((h_t)_\#(T_{\epsilon,R})\right)}\right) d\left\|\left(\phi_R^{-1}\right)_\#(h_t)_\#(T_{\epsilon,R})\right\|\\
&= \frac{d}{dt}\int_{\pi^{-1}(B(x_0,R))} E^{-1}R^{-n} f_\epsilon\left(\overrightarrow{\left((h_{t,R,E})_\#(T_\epsilon)\right)}\right) d\left\|\left((h_{t,R})_\#(T_\epsilon)\right)\right\|,
\end{aligned}$$

where $h_{t,R,E}(x, y) = (x, y + \sqrt{E}Rt\eta(x/R))$

For a given x in the good set, then for sufficiently small R this integral consists of two pieces, the integral over the good set within $B(x, R)$, which is the integral of a C^1 graph, and the integral over the bad set, which shrinks with ϵ.

Case 1. On the good set, where g_t and $G_t(X) := g_t(RX)/(\sqrt{E}R)$ are C^1, denote also by $\nabla_i G_t$ the covariant derivative of G_t in the direction of $\partial/\partial X_i$ on the ball $B(0, 1)$, with the metric stretched by the factor of $1/R$, and similarly for other maps. For maps defined on $B(0, 1)$, the notation ∇_i will refer to covariant differentiation with respect to $\partial/\partial X_i$, and for maps defined on $B(x_0, R)$, ∇_i will refer to covariant differentiation with respect to $\partial/\partial x_i$.

$$
\begin{aligned}
& \frac{d}{dt}\, \mathcal{F}_{\epsilon R}|_{B(x_0,1)\setminus\phi_R(D_\epsilon)} \left((h_t)_\# \, (T_{\epsilon,R}) \right) \\
= \; & \frac{d}{dt} \int_{\pi^{-1}(B(x_0,R)\setminus D_\epsilon)} E^{-1}R^{-n} f_\epsilon \left(\overrightarrow{\left(\phi_R^{-1}\right)_\# \left((h_t)_\# \, (T_{\epsilon,R})\right)} \right) d \left\| \left(\phi_R^{-1}\right)_\# (h_t)_\# \, (T_{\epsilon,R}) \right\| \\
= \; & \int_{\pi^{-1}(B(x_0,R)\setminus D_\epsilon)} E^{-1}R^{-n} \frac{d}{dt} \left[f_\epsilon \left(\overrightarrow{\left((h_{t,R,E})_\# \, (T_\epsilon)\right)} \right) d \left\| \left((h_{t,R,E})_\# \, (T_\epsilon)\right) \right\| \right] \\
= \; & \int_{B(x_0,R)\setminus D_\epsilon} E^{-1}R^{-n} \sum_i \frac{\sqrt{\Pi_j \left(1 + \|\nabla_j g_t\|^2\right)}}{1 + \|\nabla_i g_t\|^2} < \nabla_i g_t, \nabla_i \eta_R > d\mathcal{L}^n \\
= \; & \int_{B(0,1)\setminus\phi_R(D_\epsilon)} E^{-1} \sum_i \left. \frac{\sqrt{\Pi_j \left(1 + \|\nabla_j g_t\|^2\right)}}{1 + \|\nabla_i g_t\|^2} \right|_{x=RX} < \sqrt{E}\nabla_i G_t, \sqrt{E}\nabla_i \eta > d\mathcal{L}^n \\
= \; & \int_{B(0,1)\setminus\phi_R(D_\epsilon)} \sum_i \left. \frac{\sqrt{\Pi_j \left(1 + \|\nabla_j g_t\|^2\right)}}{1 + \|\nabla_i g_t\|^2} \right|_{x=RX} < \nabla_i G_t, \nabla_i \eta > d\mathcal{L}^n.
\end{aligned}
$$

Now, as $R \to 0$, the integral formally becomes

$$
= \int_{B(0,1)\setminus \lim_{R\to 0} \phi_R(D_\epsilon)} \sum_i \frac{\sqrt{\Pi_j \left(1 + a_j^2\right)}}{1 + A_i^2} < \nabla_i G_t, \nabla_i \eta > d\mathcal{L}^n,
$$

where a_j^2 are the critical values of the quadratic form $(v,w) \mapsto < \nabla_v g_\epsilon, \nabla_w g_\epsilon >:=< Av, w >$ as before, for unit vectors v and w, defining a linear operator A as at the end of the previous section. g_ϵ is the BV-carrier of the rectifiable section T_ϵ. The operator $\mathcal{A} = \sqrt{det(I+A)}(I+A)^{-1}$ will by elementary calculation have the same eigenvectors as A, and eigenvalues:

$$
< \mathcal{A} e_i, e_i >:= \frac{\sqrt{\Pi_j \left(1 + a_j^2\right)}}{(1 + a_i^2)}. \tag{5}
$$

Case 2. On the bad set:

LEMMA 7.1. *Given $E = Exc(T_\epsilon, R)$, and for any $\epsilon > 0$,*

$$
\left\| T_\epsilon \llcorner \pi^{-1}(D_\epsilon \cap B(x_0,R)) \right\| \;\leq\; ER^n + \epsilon A / \left|\log(\epsilon)\right|.
$$

Proof.

$$
\begin{aligned}
\left\| T_\epsilon \llcorner \pi^{-1}(D_\epsilon \cap B(x_0,R)) \right\| \;=\; & \left(\left\| T_\epsilon \llcorner \pi^{-1}(D_\epsilon \cap B(x_0,R)) \right\| - \|D_\epsilon \cap B(x_0,R)\| \right) \\
& + \|D_\epsilon \cap B(x_0,R)\| \\
\leq\; & ER^n + \|D_\epsilon \cap B(x_0,R)\| \\
=\; & ER^n + \epsilon A / \left|\log(\epsilon)\right|,
\end{aligned}
$$

where the first inequality follows from the fact that the excess is that same difference between the mass of T_ϵ and its projection (multiplicity 1) over a larger area than $D_\epsilon \cap B(x_0,R)$. □

Conversely, the excess E will give a bound on the measure of D_ϵ, which will allow us to re-estimate the mass $\left\| T_\epsilon \llcorner \pi^{-1}(D_\epsilon \cap B(x_0,R)) \right\|$ in terms only of the excess. Recall that $A \ll B$ denotes that A is bounded above by a constant times B, where that constant is independent of the variables included in A and B.

LEMMA 7.2. $\|D_\epsilon \cap B(x_0,R)\| \ll ER^n$.

Proof. On the slightly smaller set $B_\epsilon \subset D_\epsilon$, $B_\epsilon := \left\{ x \,\middle|\, h_\epsilon((\overrightarrow{T_\epsilon})_z) > 0 \, \text{for some}\, z \in \pi^{-1}(x) \right\}$, there will be at least 3 points in $\pi^{-1}(x) \cap Supp(T_\epsilon)$ for *a.e.* $x \in B_\epsilon$, because homologically $\pi_\#(T_\epsilon) = 1[B(x_0,R)]$ and, where $h_\epsilon \neq 0$, $\pi_*(\overrightarrow{T_\epsilon}) = -1\overrightarrow{\mathbb{R}^n}$, applying the constancy theorem. Thus,

$$\begin{aligned} ER^n \;&\geq\; \left\| T_\epsilon \llcorner \pi^{-1}(B_\epsilon \cap B(x_0,R)) \right\| - \|B_\epsilon \cap B(x_0,R)\| \\ &\geq\; 2\,\|B_\epsilon \cap B(x_0,R)\|\,, \end{aligned}$$

and the Lemma follows from the fact that $\|D_\epsilon \cap B(x_0,R)\| \leq 2\,\|B_\epsilon \cap B(x_0,R)\|$, by Proposition (5.1). □

Remark 7.3. On cursory examination, this Lemma would seem to imply that there is a relationship between the excess and the penalty parameter ϵ, that is, the excess could not be chosen arbitrarily small unless ϵ is itself sufficiently small. Since, however, D_ϵ can be empty independent of ϵ, that is not necessarily the case.

COROLLARY 7.4. *Given $\epsilon > 0$ and $E = Exc(T_\epsilon, R)$,*

$$\left\| T_\epsilon \llcorner \pi^{-1}(D_\epsilon \cap B(x_0,R)) \right\| \;\ll\; ER^n.$$

Proof. If $\mathbf{e}$ is the unique unit horizontal n-plane so that $\pi_*(\mathbf{e}) = *dV_M$

$$\begin{aligned} ER^n \;&:=\; \left\| T_\epsilon \llcorner \pi^{-1}(B(x_0,R)) \right\| - \|B(x_0,R)\| \\ &=\; \int_{\pi^{-1}(B(x_0,R))} \left(1 - < \overrightarrow{T_\epsilon}, \mathbf{e} >\right) d\,\|T_\epsilon\| \\ &\geq\; \int_{\pi^{-1}(D_\epsilon \cap B(x_0,R))} \left(1 - < \overrightarrow{T_\epsilon}, \mathbf{e} >\right) d\,\|T_\epsilon\| \\ &=\; \left\| T_\epsilon \llcorner \pi^{-1}(D_\epsilon \cap B(x_0,R)) \right\| - \|D_\epsilon \cap B(x_0,R)\| \\ &\gg\; \left\| T_\epsilon \llcorner \pi^{-1}(D_\epsilon \cap B(x_0,R)) \right\| - ER^n \end{aligned}$$

by Lemma (7.2). □

In addition, we have

PROPOSITION 7.5.

$$\frac{d}{dt}\, \mathcal{F}_{\epsilon R}|_{\phi_R(D_\epsilon)} \left((h_t)_\# \, (T_{\epsilon,R}) \right) \;\leq\; C\sqrt{E}.$$

Proof.

$$
\begin{aligned}
& \frac{d}{dt} \mathcal{F}_{\epsilon R}|_{\phi_R(D_\epsilon)} \left((h_t)_\# (T_{\epsilon,R})\right) \\
= & \frac{d}{dt} \int_{\pi^{-1}(D_\epsilon)} E^{-1} R^{-n} f_\epsilon \left(\overrightarrow{\left(\phi_R^{-1}\right)_\# \left((h_t)_\# (T_{\epsilon,R})\right)}\right) d \left\| \left(\phi_R^{-1}\right)_\# (h_t)_\# (T_{\epsilon,R}) \right\| \\
= & \int_{\pi^{-1}(D_\epsilon)} E^{-1} R^{-n} \frac{d}{dt} \left[f_\epsilon \left(\overrightarrow{\left((h_{t,R,E})_\# (T_\epsilon)\right)} \right) d \left\| \left((h_{t,R,E})_\# (T_\epsilon)\right) \right\| \right] \\
= & \int_{\pi^{-1}(D_\epsilon)} E^{-1} R^{-n} < \overrightarrow{T_\epsilon}, \overrightarrow{h_{t,R,E}} > d \|T_\epsilon\| \\
\le & \int_{\pi^{-1}(D_\epsilon)} E^{-1} R^{-n} \sqrt{E} d \|T_\epsilon\| \\
\le & \ C\sqrt{E}.
\end{aligned}
$$

□

8. Technical estimates

There are a number of technical estimates we will need of higher Sobolev and L^p norms for the BV carrier f of T_ϵ over $B(x_0, R)$. The notation is as in the previous section. These results are all slight modifications of results in [2]. The present situation is, unfortunately, slightly different from that considered by Bombieri, so that the statements, and proofs, need to be altered.

Following [2], first we show that

LEMMA 8.1.

$$\int_{B(x_0,R)} (\|(dx, df)\| - 1) \, d\mathcal{L}^n \le E R^n.$$

Proof. If η is smooth and of compact support in the interior of $B(x_0, R)$, then

$$
\begin{aligned}
\int \eta D_i f_j &= -\int \frac{\partial \eta}{\partial x_i} f_j \\
&= -T_\epsilon \left(y_j \frac{\partial \eta}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n \right) \\
&= T_\epsilon \left((-1)^i y_j d \left(\eta dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \right) \right) \\
&= T_\epsilon \left((-1)^i d \left(y_j \eta dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \right) \right) + \\
&\quad + T_\epsilon \left((-1)^{i-1} \eta dy_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \right) \\
&= (-1)^i \partial T_\epsilon \left(y_j \eta dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \right) + \\
&\quad + T_\epsilon \left((-1)^{i-1} \eta dy_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \right) \\
&= T_\epsilon \left((-1)^{i-1} \eta dy_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \right).
\end{aligned}
$$

Thus, by the definition of mass and the definition of f as the BV-carrier,

$$\sup \int_{B(x_0,R)} \left(\eta_0 dx_1 \wedge \cdots \wedge dx_n + \sum_{ij} \eta_{ij} D_i f_j \right) \le M(T_\epsilon \llcorner \pi^{-1}(B(x_0, R))) \tag{6}$$

where the supremum is over all (η_0, η_{ij}) of pointwise norm less than or equal to 1. Since that supremum on the left is the total variation of (dx, df), subtracting $\int_{B(x_0,R)} 1 d\mathcal{L}^n$ from both sides yields the statement. $\square$

LEMMA 8.2.

$$\int_{B(x_0,R)} \|df\| \ll \sqrt{E} R^n.$$

Thus, there is a y^* so that

$$\int_{B(x_0,R)} |f(x) - y^*| \, d\mathcal{L}^n \ll \sqrt{E} R^{n+1}.$$

Proof. In the inequality (6), set $\eta_0 = 1 - \tau$, $\tau > 0$, put all but the $D_i f_j$ terms on the right hand side, and we get

$$\int_{B(x_0,R)} \left(\sum_{ij} \eta_{ij} D_i f_j \right) \leq (\omega_n \tau + E) R^n$$

for all $\eta_{ij}$with $\sum \eta_{ij}^2 \leq 2\tau - \tau^2$, so

$$\int_{B(x_0,R)} \|df\| \leq \frac{(\omega_n \tau + E) R^n}{\sqrt{2\tau - \tau^2}}.$$

Choose $\tau = E/(E + \omega_n)$, then

$$\int_{B(x_0,R)} \|df\| \leq \sqrt{E + \omega_n} \sqrt{E} R^n.$$

The second inequality follows from the first by a Poincaré-type inequality for BV functions, proved by the standard contradiction argument using the compactness theorem for BV functions. $\square$

Remark 8.3. Note that the implicit constant in the $\ll$ of the statement of the Lemma is independent of E.

LEMMA 8.4. *For each $\epsilon > 0$, the bad set D_ϵ can be chosen so that $\|\nabla g\| \ll 1/\sqrt{E}$ on $B(x_0, R) \backslash D_\epsilon$.*

Proof. By Lemma (8.2), there is a constant C so that $\int_{B(x_0,R)} \|df\| \, d\mathcal{L}^n \leq C\sqrt{E} R^n$. For each $A > 0$, $\|\{x \in B(x_0, R) | \, \|df(x)\| > A\}\| < C\sqrt{E} R^n / A$. Given $1 > \epsilon > 0$, enlarge the bad set D_ϵ to also include $\left\{ x \in B(x_0, R) | \, \|df(x)\| > 1/\sqrt{E} \right\}$, which will still keep the measure of the bad set $\|D_\epsilon\| \ll E R^n$. $\square$

LEMMA 8.5. *For each penalty-minimizer T_ϵ, there is a $\gamma_1 > 0$ so that if the excess $E < \gamma_1$, we have*

$$Supp(T_\epsilon \llcorner \pi^{-1}(B(x_0, R'))) \subset \left\{ |y - y^*| \leq E^{\frac{1}{4n}} R \right\},$$

where $R' = (1 - E^{1/4n}) R$.

Proof. Initially, we need some basic estimates.

From Corollary [7.4], $\|D_\epsilon\| < CE$. For any given v,

$$\begin{aligned} & (vR) \cdot meas \left(B(x_0, R) \cap \{ x \notin D_\epsilon \, | \, |f(x) - y^*| \; > vR \} \right) \\ < \; & \int_{B(x_0,R)} |f(x) - y^*| \, d\mathcal{L}^n \\ \ll \; & E^{1/2} R^{n+1}, \end{aligned}$$

by (8.2) for the last inequality. Then,

$$meas\,(B(x_0,R)\cap\{x\notin D_\epsilon\,||f(x)-y^*|>vR\})\ll\frac{1}{v}E^{1/2}R^n.$$

This implies that

$$\begin{aligned}&\|T_\epsilon\|\,\{z=(x,y)\,||y-y^*|>vR\}\\ \ll\;&\|T_\epsilon\|\llcorner\pi^{-1}(D_\epsilon)+meas\,(B(x_0,R)\cap\{x\notin D_\epsilon\,|\,|f(x)-y^*|\;>vR\})+ER^n\\ \ll\;&(2ER^n+\frac{1}{v}E^{1/2})R^n.\end{aligned}$$

The proof of the Lemma now follows by a contradiction argument. Choose $v=\frac{1}{2}E^{1/4n}$ and suppose there is a $z_0\in supp(T)$, $z_0=(x_0,y_0)$, with $|y_0-y^*|>2vR$, and with $|x_0|<(1-2v)R$ (without loss of generality we can take $x_0=0$). Then,

$$\{z\,||z-z_0|\le vR\}\subset\{z=(x,y)\,|\,|y-y^*|>vR\;,\;|x-x_0|<R\}$$

and so the previous inequality implies

$$\mathcal{M}\left(T_\epsilon\llcorner\{z=(x,y)\,||z-z_0|\le vR\}\right)\ll(E+\frac{1}{v}E^{1/2})R^n$$

Now, the monotonicity result Proposition(4.1) implies that for $\epsilon>0$ sufficiently small

$$(vR)^n\ll\mathcal{M}\left(T_\epsilon\llcorner\{z\,|\,|z-z_0|\;\le vR\}\right),$$

stringing these inequalities together implies

$$\frac{1}{2^n}E^{1/4}R^n\ll(E+2E^{1/2-1/4n})R^n\ll(E+2E^{1/2-1/4n})R^n.$$

However, since the constant implied in the $\ll$ of this inequality is again independent of E, for sufficiently small E this inequality will fail. Thus, there is a sufficiently small E, $E\le\gamma_1$, for which there is no such z_0; that is, for which the statement of the Lemma will hold. □

LEMMA 8.6. *Set*

$$\overline{y}:=\frac{1}{\|B(x_0,R/2)\backslash(D_\epsilon\cap B(x_0,R/2))\|}\int_{B(x_0,R/2)\backslash(D_\epsilon\cap B(x_0,R/2))}fd\mathcal{L}^n.$$

Then

$$\int_{\pi^{-1}(B(x_0,R/2))}|y-\overline{y}|^2\,d\,\|T_\epsilon\|\ll E^{1+1/2n}R^{n+2}+\int_{B(x_0,R/2)\backslash D_\epsilon}|f-\overline{y}|^2\,d\mathcal{L}^n.$$

Proof. We have, from the proof of Lemma (8.5) that, for any $s>0$,

$$\begin{aligned}&\|T_\epsilon\|\left(\pi^{-1}(B(x_0,R/2))\cap\{|y-\overline{y}|>s\}\right)\\ \ll\;&ER^n\\ &+\mathrm{meas}(B(x_0,R/2)\backslash(D_\epsilon\cap B(x_0,R/2))\cap\{|y-\overline{y}|>s\}),\end{aligned}$$

where the first term on the right-hand side is a bound on the mass over the bad set $D_\epsilon\cap B(x_0,R/2)$. Set $Y=\sup_{y\in Supp(T\llcorner\pi^{-1}(B(x_0,R/2)))}|y-\overline{y}|$, and we have

$$\begin{aligned}&\int_{\pi^{-1}(B(x_0,R/2))}|y-\overline{y}|^2\,d\,\|T_\epsilon\|\\ =\;&2\int_0^Y s\mathcal{M}\left(T\llcorner\pi^{-1}(B(x_0,R/2))\cap\{|y-\overline{y}|>s\}\right)ds\\ \ll\;&Y^2ER^n+\int_{B(x_0,R/2)\backslash D_\epsilon}|f-\overline{y}|^2\,d\mathcal{L}^n.\end{aligned}$$

Choose $\overline{x}$ with $|\overline{x}-x_0| \le R/2$ and so that $(\overline{x},\overline{y})$ is in the convex closure of $supp(T_\epsilon \llcorner \pi^{-1}(B(x_0,R/2)))$ for ϵ sufficiently small so that the estimates in Lemma(8.5) hold. That lemma then implies that

$$Y \le \sup_{\pi^{-1}(B(x_0,R/2))} |y-y^*| + |y^*-\overline{y}| \le 2E^{1/4n}R.$$

For $\epsilon > 0$ sufficiently small, substituting this inequality in above yields the Lemma. □

The following result, unlike the others of this section, is not merely closely modeled upon the results of [2], it is precisely as given in that paper. See [2] for the proof, where it is Lemma 7.

LEMMA 8.7. *Let $0 < \theta \le 1$, $1 \le p < \frac{n}{n-1}$. there is a constant $\tau = \tau(\theta,p)$ such that if A is a measurable subset of $B(x_0,R)$, if*

$$meas(A) \ge \theta\, meas(B(x_0,R)),$$

if $h \in BV(B(x_0,R)$, and if either

$$\int_A h d\mathcal{L}^n = 0 \text{ or } \textstyle\int_A sign(h)\,|h|^{1/2}\,d\mathcal{L}^n = 0,$$

then

$$\left(R^{-n}\int_{B(x_0,R)} |h|^p\,d\mathcal{L}^n\right)^{1/p} \le \tau R^{1-n}\int_{B(x_0,R)} |Dh|\,d\mathcal{L}^n.$$

LEMMA 8.8. *For*

$$\overline{y} := \frac{1}{\|B(x_0,R/2)\backslash(D_\epsilon\cap B(x_0,R/2))\|}\int_{B(x_0,R/2)\backslash(D_\epsilon\cap B(x_0,R/2))} f d\mathcal{L}^n,$$

as in Lemma (8.6), and if $E < 1$, $1 \le p < \frac{n}{n-1}$, we have

$$\int_{B(x_0,R/2)\backslash(B(x_0,R/2)\cap D_\epsilon)} |f-\overline{y}|^{2p}\,d\mathcal{L}^n \ll_p R^{n+2p}E^{p(1+1/2n)}.$$

Proof. We may assume that $\overline{y}=0$. For $\phi = \phi(x)dx_1\wedge\cdots\wedge dx_n$ a horizontal form, define currents V_j by

$$V_j(\phi) := T_\epsilon(y_j|y_j|\phi)$$

and represent it by integration as

$$V_j(\phi) = \int_{B(x_0,R)} h_j(x)\phi$$

with $h_j \in BV(B(x_0,R))$. By the definition of the good set $B(x_0,R/2)\backslash(B(x_0,R/2)\cap D_\epsilon)$, $h_j = f_j|f_j|$ on the good set. If $\psi = \sum_i \psi_i dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n$ is smooth, with compact support in the interior of $B(x_0,R)$, we have

$$\begin{aligned}\partial V_j(\psi) &= T_\epsilon(y_j|y_j|d\psi)\\ &= \partial T_\epsilon(y_j|y_j|\psi) - 2T_\epsilon(|y_j|dy_j\wedge\psi)\\ &= -2T_\epsilon(|y_j|dy_j\wedge\psi).\end{aligned}$$

If ψ has compact support within $B(x_0, R/2)$,

$$
\begin{aligned}
|\partial V_j(\psi)| &\leq 2\int_{B(x_0,R/2)} |y_j| \sum_i |\psi_i| \left| \left\langle dy_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n, \overrightarrow{T_\epsilon} \right\rangle \right| d\,\|T_\epsilon\| \\
&\leq 2\,(\sup \|\psi\|) \int_{B(x_0,R/2)} |y_j| \left(\sum_i \left| \left\langle dy_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n, \overrightarrow{T_\epsilon} \right\rangle \right|^2 \right)^{1/2} d\,\|T_\epsilon\| \\
&\leq 2\,(\sup \|\psi\|) \left(\int_{C(x_0,R/2)} |y_j|^2 \, d\,\|T_\epsilon\| \right)^{1/2} \left(\int_{C(x_0,R/2)} \left[1 - \left| \left\langle dx, \overrightarrow{T_\epsilon} \right\rangle \right|^2 \right] d\,\|T_\epsilon\| \right)^{1/2} \\
&\leq 2\,(\sup \|\psi\|) \left(\int_{C(x_0,R/2)} |y_j|^2 \, d\,\|T_\epsilon\| \right)^{1/2} (2ER^n)^{1/2} .
\end{aligned}
$$

Lemma (8.6) and this inequality implies that

$$
\begin{aligned}
\int_{B(x,R/2)} |Dh_j| \, d\mathcal{L}^n &= M\left((\partial V_j) \llcorner B(x_0, R/2) \right) \\
&\ll \left(\int_{C(x_0,R/2)} |y_j|^2 \, d\,\|T_\epsilon\| \right)^{1/2} (2ER^n)^{1/2} \\
&\ll (2ER^n)^{1/2} \left(E^{1+1/2n} R^{n+2} + \int_{B(x_0,R/2)\setminus D_\epsilon} |h_j| \, d\mathcal{L}^n \right)^{1/2} .
\end{aligned}
$$

Now apply Lemma(8.7) with $A := B(x_0, R/2)\setminus D_\epsilon$ inside of $B(x_0, R/2)$, which implies that

$$\int_A |h_j| \, d\mathcal{L}^n \ll R \int_{B(x_0,R/2)} |Dh_j| \, d\mathcal{L}^n.$$

Combining this with the previous inequality,

$$\left(\int_{B(x_0,R/2)} |Dh_j| \, d\mathcal{L}^n \right)^2 \ll (2ER^n) \left(E^{1+1/2n} + R \int_{B(x_0,R/2)} |Dh_j| \, d\mathcal{L}^n \right),$$

which by the quadratic formula and the fact that $E < 1$ implies that

$$\int_{B(x_0,R/2)} |Dh_j| \, d\mathcal{L}^n \ll E^{1+1/2n} R^{n+1}.$$

Applying Lemma (8.7) gives the statement. □

LEMMA 8.9. *There is an r with $R/4 \leq r \leq R/3$, for which, given $0 < \mu \leq 1$, there is a current S so that*

1. $\partial(S \llcorner C(x_0, r)) = \partial(T_\epsilon \llcorner C(x_0, r))$,
2. $\partial(\pi_\#(S \llcorner C(x_0, R)) = \partial B(x_0, R)$,
3. $diam(Supp(S \llcorner C(x_0, R)) \cup Supp(T_\epsilon \llcorner C(x_0, r))) \leq R$,
4. $Exc(S, R) \ll \mu E + E^{1+1/2n}/\mu + \int_{B(x_0,R/2)\setminus D_\epsilon} |f - \overline{y}|^2 \, d\mathcal{L}^n \Big/ (\mu R^{n+2})$.

Proof. As before, normalize so that $\overline{y} = 0$. If S is any normal current in $\Omega \times \mathbb{R}^k$, the slice

$$< S, r > := \partial(S \llcorner C(x_0, r)) - (\partial S) \llcorner C(x_0, r)$$

satisfies, for smooth functions g,

$$< S, r > \llcorner g = < S \llcorner g, r >$$

for almost every r, where $S \llcorner g(\phi) := S(g\phi)$, and

$$\int_0^p \mathcal{M}(< S, r >) dr \leq \mathcal{M}(S \llcorner C(x_0, p))$$

(cf. Morgan, p. 55). Applied to T_ϵ, with $g = |y|^2$, and $p = R/2$, we have

$$\begin{aligned} \int_0^{R/2} \left(\mathcal{M}(< T_\epsilon, r >) - n\alpha_n r^{n-1}\right) dr &\leq \mathcal{M}(T_\epsilon \llcorner C(x_0, R/2)) - \alpha_n R^n / 2^n \\ &\leq \left(\frac{R}{2}\right)^n Exc(T_\epsilon, R/2) \\ &\leq R^n E, \end{aligned} \tag{7}$$

and, from Lemma (8.6)

$$\begin{aligned} \int_0^{R/2} \mathcal{M}(< T_\epsilon, r > \llcorner |y|^2) dr &= \int_0^{R/2} \mathcal{M}(< T_\epsilon \llcorner |y|^2, r >) dr \\ &\leq \int_{C(x_0, R/2)} |y|^2 d \, \|T_\epsilon\| \\ &\leq E^{1+1/2n} R^{n+2} + \int_{B(x_0, R/2) \setminus D_\epsilon} |f|^2 \, d\mathcal{L}^n. \end{aligned} \tag{8}$$

Note also that

$$\mathcal{M}(< T_\epsilon, r >) - n\alpha_n r^{n-1} \geq \mathcal{M}(< \pi_\#(T_\epsilon), r >) - n\alpha_n r^{n-1} = 0,$$

since $\pi_\#$ is mass-decreasing.

From (7), there is some r with

$$\mathcal{M}(< T_\epsilon, r >) - n\alpha_n r^{n-1} \ll E R^{n-1},$$

and due to the implicit constant in the inequality, such an r can be found in $[R/4, R/3]$. We can also find, using (8), a choice of $r \in [R/4, R/3]$ also satisfying

$$\mathcal{M}(< T_\epsilon, r > \llcorner |y|^2) \ll E^{1+1/2n} R^{n+1} + \frac{1}{R} \int_{B(x_0, R/2) \setminus D_\epsilon} |f|^2 \, d\mathcal{L}^n.$$

We now construct a comparison current. Set S to be the current

$$S := B(x_0, (1-\mu)r) \times \{0\} + h_\#([1-\mu, 1+\mu] \times < T_\epsilon, r >) + (B(x_0, R) - B(x_0, (1+\mu)r)) \times \{0\},$$

where

$$h(t, x, y) = (tx, y - |t-1| y/\mu).$$

S is a deformation of the horizontal current $B(x_0, R) \times \{0\}$ that matches with the slice of T_ϵ at radius r, but which is still flat off of an annulus of width 2μ. It is clear from the construction that this current satisfies (1) and (2) of the statement.

Since $|\partial h/\partial t| \leq (r^2 + |y|^2/\mu^2)^{1/2}$ (also cf. [5, 4.1.9])

$$\mathcal{M}(h_\#([1-\mu, 1+\mu]\times < T_\epsilon, r >)) \leq \int_{1-\mu}^{1+\mu} t^{n-1} \int (r^2 + |y|^2/\mu^2)^{1/2} d\,\| < T_\epsilon, r > \|\, dt.$$

Performing the indicated integration with respect to t and noting that $(r^2+|y|^2/\mu^2)^{1/2} \leq \left(r + \frac{|y|^2}{\mu^2 r}\right)$,

$$\begin{aligned} \mathcal{M}(h_\#([1-\mu, 1+\mu]\times < T_\epsilon, r >)) &\leq \left(\frac{(1+\mu)^n - (1-\mu)^n}{n}\right) \int \left(r + \frac{|y|^2}{\mu^2 r}\right) d\,\| < T_\epsilon, r > \| \\ &\leq (2n\mu)\left(r\mathcal{M}(< T_\epsilon, r >) + \frac{1}{\mu^2 r}\mathcal{M}(< T_\epsilon, r > \llcorner |y|^2)\right). \quad (9) \end{aligned}$$

Now,

$$\begin{aligned} Exc(S, R) &= \mathcal{M}(S \llcorner C(x_0, R))/R^n - \alpha_n \\ &= \mathcal{M}(h_\#([1-\mu, 1+\mu]\times < T_\epsilon, r >))/R^n + \alpha_n((1-\mu)^n r^n + (1+\mu)^n r^n)/R^n \\ &\leq 2n\mu\left(r\mathcal{M}(< T_\epsilon, r >) + \frac{1}{\mu^2 r}\mathcal{M}(< T_\epsilon, r > \llcorner |y|^2)\right)/R^n + \alpha_n(-2n\mu r^n)/R^n \\ &\ll \mu\left(r(ER^{n-1}) + \frac{1}{\mu^2 r}\left(E^{1+1/2n}R^{n+1} + \frac{1}{R}\int_{B(x_0,R/2)\setminus D_\epsilon} |f-\overline{y}|^2\, d\mathcal{L}^n\right)\right)/R^n \\ &\ll \mu E + \frac{1}{\mu}E^{1+1/2n} + \frac{1}{\mu R^{n+2}}\left(\int_{B(x_0,R/2)\setminus D_\epsilon} |f-\overline{y}|^2\, d\mathcal{L}^n\right), \end{aligned}$$

which is part (4) of the Lemma.

Part (3) of the Lemma follows from Lemma (8.5). □

LEMMA 8.10. *If $R \leq \gamma_3$, then, for $0 < \mu \leq 1$ chosen as before, and if $E \leq \min\{\gamma_1, (2/3)^{4n}\}$,*

$$Exc(T_\epsilon, R/4) \ll \mu E(1 + \frac{1}{2\epsilon}) + E\left(\frac{E^{1/2n}}{\mu}(1 + \frac{1}{2\epsilon})\right) + \int_{B(x_0,R/2)\setminus D_\epsilon} |f-\overline{y}|^2\, d\mathcal{L}^n \Big/ (\mu R^{n+2}).$$

Proof. Again, suppose that $\overline{y} = 0$. Let S be as in Lemma (8.9). and set

$$\widetilde{T} := T_\epsilon \llcorner C(x_0, r) + S - S \llcorner C(x_0, r),$$

which replaces T_ϵ by S outside of the cylinder of radius r, without introducing any interior boundaries by the construction of S. Note that $\partial\widetilde{T} = \partial B(x_0, R) \times \{0\}$. By construction, monotonicity of the unnormalized excess, and the choice of r, $R/4 \leq r \leq R/3$,

$$(R/4)^n Exc(T_\epsilon, R/4) \leq r^n Exc(T_\epsilon, r) = r^n Exc(\widetilde{T}, r) \leq R^n Exc(\widetilde{T}, R).$$

By the definition of the penalty functional,

$$\begin{aligned} Exc(\widetilde{T}, R) &:= \left(\mathcal{M}(\widetilde{T}) - \mathcal{M}(B(x_0, R) \times \{0\})\right)/R^n \\ &\leq \left(\mathcal{F}_\epsilon(\widetilde{T}) - \mathcal{F}_\epsilon(B(x_0, R) \times \{0\})\right)/R^n. \end{aligned}$$

Using minimality,

$$\mathcal{F}_\epsilon(T_\epsilon \llcorner C(x_0, r)) \leq \mathcal{F}_\epsilon(S \llcorner C(x_0, r)),$$

so that

$$\begin{aligned}\mathcal{F}_\epsilon(\widetilde{T}) &= \mathcal{F}_\epsilon(T_\epsilon \llcorner C(x_0,r) + (S - S \llcorner C(x_0,r))) \\ &\leq \mathcal{F}_\epsilon(S).\end{aligned}$$

Thus,

$$\begin{aligned}Exc(\widetilde{T},R) &\leq \left(\mathcal{F}_\epsilon(\widetilde{T}) - \mathcal{F}_\epsilon(B(x_0,R)\times\{0\})\right)/R^n \\ &\leq \left(\mathcal{F}_\epsilon(S) - \mathcal{M}(B(x_0,R)\times\{0\})\right)/R^n.\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{F}_\epsilon(S) &= \mathcal{M}(B(x_0,(1-\mu)r)\times\{0\}) + \mathcal{M}((B(x_0,R) - B(x_0,(1+\mu)r))\times\{0\}) \\ &\quad + \mathcal{F}_\epsilon(h_\#([1-\mu,1+\mu]\times < T_\epsilon, r >)),\end{aligned}$$

and the slice $< T_\epsilon, r >$ is the graph of the C^1 function f on $\partial B(x_0,r)\backslash(D_\epsilon \cap \partial B(x_0,r))$. The integral over the bad set $D_\epsilon \cap \partial B(x_0,r)$ will, for some $r \in [R/4, R/3]$ consistent with all previous choices of r, be bounded by the mass over that set plus $(12/R)(ER^n)(\frac{1}{2\epsilon}) = 12R^{n-1}(E)(\frac{1}{2\epsilon})$ by Corollary (7.4) and the definition of $\mathcal{F}_\epsilon$. So, similarly to equation (9)the proof of Lemma (8.9), but using the height bound of Lemma (8.5) to bound $|y|$, along with the estimate for $|y|$ from Lemma (8.6),

$$Supp(T_\epsilon \llcorner \pi^{-1}(B(x_0,R'))) \subset \left\{|y-y^*| \leq E^{\frac{1}{4n}}R\right\},$$

and

$$\sup_{\pi^{-1}(B(x_0,R/2))} |y-y^*| + |y^* - \overline{y}| \leq 2E^{1/4n}R.,$$

with $\overline{y} = 0$, implying that $|y| < 2E^{1/4n}R$, to bound the contribution from the sloped sides of S on the bad set,

$$\begin{aligned}\mathcal{F}_\epsilon(h_\#([1-\mu,1+\mu]\times < T_\epsilon,r >)) &\leq \int_{B(x_0,r(1+\mu))\backslash B(x_0,r(1-\mu))} \mathcal{M}(h_\#([1-\mu,1+\mu]\times < T_\epsilon,r >)) \\ &+ (r^2 + |y|^2/\mu^2)^{1/2}\left(\frac{(1+\mu)^n - (1-\mu)^n}{n}\right)(12R^{n-1})(E)(\frac{1}{2\epsilon}) \\ &\leq \int_{B(x_0,r(1+\mu))\backslash B(x_0,r(1-\mu))} \mathcal{M}(h_\#([1-\mu,1+\mu]\times < T_\epsilon,r >)) \\ &\quad +2n\mu\left(r + \frac{2R^2E^{1/2n}}{\mu^2 r}\right)(12R^{n-1})(E)(\frac{1}{2\epsilon}).\end{aligned}$$

Combining this inequality with Lemma (8.9),

$$\begin{aligned}Exc(T_\epsilon, R/4) &\leq 4^n Exc(\widetilde{T},R) \\ &\leq 4^n\left(\mathcal{F}_\epsilon(S) - \mathcal{M}(B(x_0,R)\times\{0\})\right)/R^n \\ &\leq 4^n\,(\mathcal{M}(S) - \mathcal{M}(B(x_0,R)\times\{0\}) \\ &\quad +2n\mu\left(r + \frac{2R^2E^{1/2n}}{\mu^2 r}\right)(12R^{n-1})(E)(\frac{1}{2\epsilon})\Bigg)/R^n \\ &\leq 4^n\left(Exc(S) + 2n\mu\left(r + \frac{2R^2E^{1/2n}}{\mu^2 r}\right)(12R^{-1})(E)(\frac{1}{2\epsilon})\right) \\ &\ll \mu E(1+\frac{1}{2\epsilon}) + E\left(\frac{E^{1/2n}}{\mu}(1+\frac{1}{2\epsilon})\right) + \int_{B(x_0,R/2)\backslash D_\epsilon} |f-\overline{y}|^2\, d\mathcal{L}^n \Big/ (\mu R^{n+2}),\end{aligned}$$

as required. $\square$

9. First variation of $\mathcal{F}_\epsilon(T)$

Consider the deformations $(h_t)_\#(T_\epsilon)$ of T_ϵ, where h_t is given by

$$h_t(x,y) := (x, y + t\sqrt{E}R\eta(x/R)),$$

for $-1 < t < 1$, and η smooth with compact support in $|X| < 1$, with $\|\nabla\eta\| \leq \beta$. Given the blow-up map

$$\phi_R(x,y) = \left(\frac{x}{R}, \frac{y}{\sqrt{E}R}\right) := (X, Y),$$

define $F : B(x_0, 1) \to \mathbb{R}^j$ by

$$F(X) = f(RX)/(\sqrt{E}R),$$

where f is, as before, the BV-carrier of T_ϵ. On the good set, moreover, $G_\epsilon(X) = g_\epsilon(RX)/\sqrt{E}R$, and so $\nabla_X G_\epsilon = \frac{1}{\sqrt{E}}\nabla_x g_\epsilon$, where g_ϵ is the graph representing T_ϵ on the good set.

LEMMA 9.1. *If $\eta(X)$ is smooth with compact support in $|X| < 1$, $|\nabla\eta| \leq 1$, then given a deformation h_t given by*

$$h_t(x,y) := (x, y + t\sqrt{E}R\eta(x/R))$$

and if $T_{\epsilon,R} = (\phi_R)_\#(T_\epsilon)$, where $\phi_R(x,y) = (x/R, y/(\sqrt{E}R))$, then

$$\left|\frac{d}{dt}\mathcal{F}_{\epsilon,R}\left((h_t)_\#\left(T_{\epsilon,R}\right)\right) - \int_{B(X_0,1)} \sum_{i,k} \mathcal{A}_{ik} \left\langle \nabla_i\eta, \nabla_k F + t\nabla_k\eta \right\rangle d\mathcal{L}^n\right| \ll \sqrt{E}.$$

Proof. By Lemma (7.4), and the definition of the bad set D_ϵ in Proposition (5.1), we find a C^1 function $g_\epsilon : B(x_0, R)\backslash D_\epsilon \to F$ whose graph agrees with T_ϵ over $B(x_0, R)\backslash D_\epsilon$, and $g_t(x) := g_\epsilon(x) + t\sqrt{E}R\eta(x/R)$. Then

$$\begin{aligned} L(t) &:= \mathcal{F}_\epsilon((h_t)_\#(graph(g_\epsilon))\llcorner(C(x_0,R)\backslash\pi^{-1}(D_\epsilon))/(ER^n), \\ K(t) &:= \mathcal{F}_\epsilon((h_t)_\#(T_\epsilon)\llcorner(C(x_0,R)\cap\pi^{-1}(D_\epsilon))/(ER^n) \end{aligned}$$

so that

$$\mathcal{F}_\epsilon((h_t)_\#(T_\epsilon))/(ER^n) = L(t) + K(t).$$

Apply the squash-deformation $\phi_R(x,y) := (x/R, y/(\sqrt{E}R))$. If $T_{\epsilon,R} := (\phi_R)_\#(T_\epsilon\llcorner C(x_0,R))$, it will minimize the functional $\mathcal{F}_{\epsilon,R}$ defined by

$$\mathcal{F}_{\epsilon,R}(S) := \mathcal{F}_\epsilon((\phi_R^{-1})_\#(S))/(ER^n),$$

so that, on $T_{\epsilon,R}$, $\mathcal{F}_{\epsilon,R}(T_{\epsilon,R}) := \mathcal{F}_\epsilon(T_\epsilon\llcorner C(x_0,R))/(ER^n)$. Explicitly, for S a graph on $\pi^{-1}(\Omega) \subset C(X_0, 1)$,

$$\mathcal{F}_{\epsilon,R}(S) \;=\; \left\|(\phi_R^{-1})_\#(S)\right\|/(ER^n) + \frac{1}{\epsilon ER^n}\mathcal{H}_0(S),$$

where $\mathcal{H}_0$ is as defined in the beginning of §4.

On the good set, since the penalty term vanishes there,

$$\begin{aligned}\frac{d}{dt}L(t) &= \frac{d}{dt}\mathcal{F}_\epsilon((h_t)_\#(graph(g_\epsilon))\llcorner(C(x_0,R)\backslash\pi^{-1}(D_\epsilon))/(ER^n) \\ &= \frac{d}{dt}\mathcal{M}((h_t)_\#(graph(g_\epsilon))\llcorner(C(x_0,R)\backslash\pi^{-1}(D_\epsilon))/(ER^n) \\ &= \int_{\pi^{-1}(B(x_0,R)\backslash D_\epsilon)}\sum_i \frac{<\nabla_i g_t, \nabla_i h>}{1+\|\nabla_i g_t\|^2} d\,\|T_t\| \Big/ (ER^n)\end{aligned}$$

by [6.1]. Since $g_t(x) := g_\epsilon(x) + t\sqrt{E}R\eta(x/R)$ and $h(x) = \frac{dg_t}{dt} = \sqrt{E}R\eta(x/R)$,

$$\begin{aligned}\frac{d}{dt}L(t) &= \int_{\pi^{-1}(B(x_0,R)\backslash D_\epsilon)} \sqrt{E}\sum_i \frac{<\nabla_i g_\epsilon, \nabla_i\eta> + t\sqrt{E}<\nabla_i\eta,\nabla_i\eta>}{1+\|\nabla_i g_t\|^2} d\,\|T_t\|/ER^n \\ &= \int_{B(x_0,R)\backslash D_\epsilon} \frac{1}{\sqrt{E}R^n}\sum_i \frac{<\nabla_i g_\epsilon, \nabla_i\eta> + t\sqrt{E}<\nabla_i\eta,\nabla_i\eta>}{1+\|\nabla_i g_\epsilon\|^2 + 2t\sqrt{E}<\nabla_i g_\epsilon,\nabla_i\eta> + t^2E\,\|\nabla_i\eta\|^2} \\ &\quad \left\|(e_1+\nabla_1 g_\epsilon + t\sqrt{E}\nabla_1\eta)\wedge\cdots\wedge(e_n+\nabla_n g_\epsilon + t\sqrt{E}\nabla_n\eta)\right\| d\mathcal{L}^n.\end{aligned}$$

Now apply the squash-deformation $\phi_R(x,y) = (X,Y) := (x/R, y/(\sqrt{E}R))$. Explicitly, for S a graph, $S = graph(P(X))$ on $C(X_0,1)$,

$$\begin{aligned}\mathcal{F}_{\epsilon,R}(S) &= \left\|(\phi_R^{-1})_\#(S)\right\|/(ER^n) + \frac{1}{\epsilon ER^n}H_0(S) \\ &= \frac{1}{L}\int_{B(x_0,R)}\sqrt{1+E\,\|\nabla_i P\|^2 + \cdots + E^n\,\|\nabla_{i_1}P\wedge\cdots\wedge\nabla_{i_n}P\|^2}d\mathcal{L}^n,\end{aligned}$$

keeping in mind that the penalty term vanishes on graphs. Use a coordinate system $\{x^1,\ldots,x^n\}$ so that the quadratic form $A_\epsilon(v,w) \mapsto <\nabla_v g_\epsilon, \nabla_w g_\epsilon>|_{x_0}$ is diagonalized, with eigenvalues a_i^2. The operator $\mathcal{A}_\epsilon := \sqrt{det(I+A_\epsilon)}(I+A_\epsilon)^{-1}$ with the same eigenvectors but with eigenvalues $\mathcal{A}_{\epsilon,i} = \frac{\sqrt{\Pi_j(1+a_j^2)}}{1+a_i^2}$ is the first term in the expansion of the previous expression.

$$\begin{aligned}\frac{d}{dt}L(t) &= \frac{d}{dt}\mathcal{F}_\epsilon((h_t)_\#(graph(g_\epsilon))\llcorner(C(x_0,R)\backslash\pi^{-1}(D_\epsilon))/(ER^n) \\ &= \\ &= \frac{d}{dt}\mathcal{F}_{\epsilon,R}((\phi_R)_\#(h_t)_\#(T_\epsilon)\llcorner C(x_0,R)\backslash\pi^{-1}(D_\epsilon)) \\ &= \frac{d}{dt}\mathcal{F}_{\epsilon,R}\left(graph(G_\epsilon+t\eta)\llcorner\left(C(X_0,1)\backslash\phi_R(\pi^{-1}(D_\epsilon))\right)\right) \\ &= \frac{1}{E}\int_{B(X_0,1)\backslash\phi_R(D_\epsilon)}\sum_i\frac{E<\nabla_i G_\epsilon,\nabla_i\eta> + tE<\nabla_i\eta,\nabla_i\eta>}{1+E\,\|\nabla_i(G_\epsilon+t\eta)\|^2}\cdot \\ &\quad \cdot\sqrt{1+E\,\|\nabla(G_\epsilon+t\eta)\|^2+\cdots+E^n\,\|\nabla(G_\epsilon+t\eta)\wedge\cdots\wedge\nabla(G_\epsilon+t\eta)\|^2}d\mathcal{L}^n \\ &= \int_{B(X_0,1)\backslash\phi_R(D_\epsilon)}\sum_i(<\nabla_i G_\epsilon,\nabla_i\eta> + t<\nabla_i\eta,\nabla_i\eta>)\,\mathcal{A}_{ii}d\mathcal{L}^n + Q,\end{aligned}$$

where the coordinate basis $\{X_1,\ldots,X_n\}$ is chosen at each point to be an orthonormal eigenbasis of $(V,W)\mapsto<\nabla_V(G_\epsilon+t\eta),\nabla_W(G_\epsilon+t\eta)>$ and, at each point, $\nabla_i := \nabla_{\partial/\partial X_i}$. Since $\{\nabla_j(G_\epsilon+t\eta)\}$ is orthogonal by choice of basis,

$$\begin{aligned}&\sqrt{1+E\,\|\nabla(G_\epsilon+t\eta)\|^2+\cdots+E\,\|\nabla(G_\epsilon+t\eta)\wedge\cdots\wedge\nabla(G_\epsilon+t\eta)\|^2} \\ =\ &\sqrt{\Pi_j\left(1+E\,\|\nabla_i(G_\epsilon+t\eta)\|^2\right)}\end{aligned}$$

Choose $\{V_i\}$ to be an eigenbasis of $\mathcal{A}$ as above, that is, an eigenbasis of $(V,W) \mapsto < \nabla_V(G_\epsilon + 0\eta), \nabla_W(G_\epsilon + 0\eta) >$ at X_0.

Q is given simply as

$$\begin{aligned} Q &:= \int_{B(X_0,1)\setminus\phi_R(D_\epsilon)} \sum_i \frac{< \nabla_i G_\epsilon, \nabla_i \eta > + t < \nabla_i \eta, \nabla_i \eta >}{1 + E\left\|\nabla_i(G_\epsilon + t\eta)\right\|^2} \sqrt{\Pi_j \left(1 + E\left\|\nabla_j(G_\epsilon + t\eta)\right\|^2\right)} \\ &\quad - (< \nabla_i G_\epsilon, \nabla_i \eta > + t < \nabla_i \eta, \nabla_i \eta >)\, \mathcal{A}_{ii} d\mathcal{L}^n \\ &:= \int_{B(X_0,1)\setminus\phi_R(D_\epsilon)} \sum_i (< \nabla_i G_\epsilon, \nabla_i \eta > + t < \nabla_i \eta, \nabla_i \eta >)\, Q_i d\mathcal{L}^n. \end{aligned}$$

If

$$Q_i(P_1, \ldots, P_n) := \frac{\sqrt{\Pi_{j\neq i}\left(1 + E\left\|P_j\right\|^2\right)}}{\sqrt{1 + E\left\|P_i\right\|^2}} - \mathcal{A}_{ii},$$

$Q_i := Q_i(\nabla_1(G_\epsilon + t\eta), \ldots, \nabla_n(G_\epsilon + t\eta))$, then by a simple application of the mean value theorem at each x, there is a $c := c(x) \in (0,1)$ for which, since if $\nabla_{V_i} G_\epsilon|_{x_0} := A_i$, $Q_i(A_1, \ldots, A_n) = 0$,

$$\begin{aligned} Q_i &= \frac{\partial Q_i}{\partial P_j}(P_1(c), \ldots, P_n(c))(\nabla_j(G_\epsilon + t\eta) - A_j) \\ &= \sum_{j\neq i} E \frac{\sqrt{\Pi_{k\neq i,j}\left(1 + E\left\|P_k(c)\right\|^2\right)}}{\sqrt{1 + E\left\|P_i(c)\right\|^2}\sqrt{1 + E\left\|P_j(c)\right\|^2}} < P_j(c), \nabla_j(G_\epsilon + t\eta) - A_j > \\ &\quad - E \frac{\sqrt{\Pi_{j\neq i}\left(1 + E\left\|P_j(c)\right\|^2\right)}}{\left(1 + E\left\|P_i(c)\right\|^2\right)^{3/2}} < P_i(c), \nabla_i(G_\epsilon + t\eta) - A_i > \end{aligned}$$

for some $(P_1(c), \ldots, P_n(c)) = (A_1, \ldots, A_n) + c(\nabla_1(G_\epsilon + t\eta) - A_1, \ldots, \nabla_n(G_\epsilon + t\eta) - A_n)$, $c \in (0,1)$. Now, $f(t) = t/\sqrt{1+t}$ is increasing for $t > 0$ and $\|P_l(c)\| \ll \|\nabla_l G_\epsilon\| \ll \|P_l(c)\|$ (which follows because $\|\nabla\eta\|$ and A_l are bounded), so that

$$\sqrt{\Pi_{k\neq i,j}\left(1 + E\left\|P_k(c)\right\|^2\right)} \ll \sqrt{\Pi_{k\neq i,j}\left(1 + E\left\|\nabla_k G_\epsilon\right\|^2\right)}$$

and

$$\frac{E < P_j(c), \nabla_j(G_\epsilon + t\eta) - A_j >}{\sqrt{1 + E\left\|P_j(c)\right\|^2}} \ll \frac{E\left\|P_j(c)\right\|^2}{\sqrt{1 + E\left\|P_j(c)\right\|^2}} \ll \frac{E\left\|\nabla_j G_\epsilon\right\|^2}{\sqrt{1 + E\left\|\nabla_j G_\epsilon\right\|^2}}.$$

Then, applying these inequalities to the expression for Q_i above,

$$|Q_i| \ll \sum_j \frac{E\left\|\nabla_j G_\epsilon\right\|^2 \sqrt{\Pi_{k\neq i,j}\left(1 + E\left\|\nabla_k G_\epsilon\right\|^2\right)}}{\sqrt{1 + E\left\|\nabla_j G_\epsilon\right\|^2}\sqrt{1 + E\left\|P_i(c)\right\|^2}},$$

and so, this time because $f(t) = t/\sqrt{1+t^2}$ is also increasing, and using Lemma (8.4) in the second step,

$$|Q| \ll \frac{1}{\sqrt{E}} \int_{B(X_0,1)\setminus\phi_R(D_\epsilon)} \sum_{i,j} \frac{\sqrt{E}\left\|\nabla_i G_\epsilon\right\| E\left\|\nabla_j G_\epsilon\right\|^2 \sqrt{\Pi_{k\neq i,j}\left(1 + E\left\|\nabla_k G_\epsilon\right\|^2\right)}}{\sqrt{1 + E\left\|\nabla_i G_\epsilon\right\|^2}\sqrt{1 + E\left\|\nabla_j G_\epsilon\right\|^2}} d\mathcal{L}^n$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{E}} \int_{B(X_0,1)\setminus\phi_R(D_\epsilon)} \sum_{i,j} \frac{\|\nabla_i g_\epsilon\| \, \|\nabla_j g_\epsilon\|^2 \sqrt{\Pi_{k\neq i,j}\left(1+\|\nabla_k g_\epsilon\|^2\right)}}{\sqrt{1+\|\nabla_i g_\epsilon\|^2}\sqrt{1+\|\nabla_j g_\epsilon\|^2}}\Bigg|_{x=RX} d\mathcal{L}^n(X) \\
&\ll \frac{1}{\sqrt{E}} \int_{B(X_0,1)\setminus\phi_R(D_\epsilon)} \left(\sqrt{\Pi_k\left(1+\|\nabla_k g_\epsilon\|^2\right)}\Bigg|_{x=RX} - 1\right) d\mathcal{L}^n(X) \\
&= \sqrt{E}.
\end{aligned}
$$

The last inequality follows from the fact that

$$
4(\sqrt{1+a^2}\sqrt{1+b^2}c-1) - \frac{ab^2c}{\sqrt{1+a^2}\sqrt{1+b^2}} \geq 0
$$

for any $c > 1$, which is a straightforward calculation.

On the bad set D_ϵ, by the strong approximation theorem [5, 4.2.20] we can assume without loss of generality that $T_\epsilon \llcorner \pi^{-1}(D_\epsilon)$ is the image $\psi_\#(P)$, where P is a polyhedral chain and ψ is Lipschitz. The definition of $K(t)$ and the fact that the deformation h_t is vertical [cf. (6.1)] implies that

$$
\begin{aligned}
\frac{d}{dt}K(t) &= \frac{d}{dt}\bigg|_t \int_{\pi^{-1}(D_\epsilon)} f_\epsilon(\overrightarrow{T_t}) d\|T_t\| / (ER^n) \\
&= \int_{\pi^{-1}(D_\epsilon)} \frac{d}{dt} d\|T_t\| / (ER^n)
\end{aligned}
$$

since the deformation will leave the penalty part fixed. In addition, the derivative of this integrand will be 0 at all points with a vertical tangent plane, again due to the fact that the deformation is vertical. At all points where the tangent plane is not vertical, the mean-value theorem approximation used for the good set will again hold, where we can replace $g_\epsilon(x)$ by $\psi(p)$, where $\pi(\psi(p)) = x$. In the notation above, if

$$
\mathcal{F}_{\epsilon,R}(S) := \left\|(\phi_R^{-1})_\#(S)\right\| / (ER^n) + \frac{1}{\epsilon ER^n} H(S),
$$

then applying the squash-deformation, for which $\phi_R\psi := \Psi$

$$
\begin{aligned}
\frac{d}{dt}K(t) &= \frac{d}{dt}\mathcal{F}_{\epsilon,R}\left((H_t)_\#(\Psi_\#(P))\llcorner\left(\phi_R(\pi^{-1}(D_\epsilon))\right)\right) \\
&= \frac{1}{ER^n}\frac{d}{dt}\int_P \sqrt{\sum_{|\alpha|+|\beta|=n} E^{|\beta|}((H_t)_\#(\Psi_\#(P))_{\alpha\beta})^2} d\|P\|,
\end{aligned}
$$

where again the penalty part is irrelevant since the deformation is vertical, and the deformation H_t defined by $H_t = \phi_R h_t \phi_R^{-1}$ becomes translation vertically by $t\eta(X)$, where $X = \pi(p)$, $p \in Supp(P)$. Also as a consequence of the verticality of the deformation, the $\beta = 0$ term of the integral will be unchanged under the deformation, so

$$
\begin{aligned}
\frac{d}{dt}K(t) &= \frac{d}{dt}\mathcal{F}_{\epsilon,R}\left((H_t)_\#(\Psi_\#(P))\llcorner\left(\phi_R(\pi^{-1}(D_\epsilon))\right)\right) \\
&= \frac{1}{ER^n}\int_P \frac{E\sum_{|\alpha|+|\beta|=n,\beta\neq 0} E^{|\beta|-1}(H_t)_\#(\Psi_\#(P))_{\alpha\beta}\frac{d}{dt}(H_t)_\#(\Psi_\#(P))_{\alpha\beta}}{\sqrt{\sum_{|\alpha|+|\beta|=n} E^{|\beta|}((H_t)_\#(\Psi_\#(P))_{\alpha\beta})^2}} d\|P\|(p) \\
&= \frac{1}{R^n}\left(\int_{D_\epsilon}\sum_i \left(<\nabla_i F, \nabla_i \eta> + t <\nabla_i\eta, \nabla_i\eta>\right)\mathcal{A}_{ii} d\mathcal{L}^n + Q\right).
\end{aligned}
$$

The factors Q_i, $Q = \int_{D_\epsilon} \sum_i \left(< \nabla_i \eta, \nabla_i F + t\nabla_i \eta >\right) Q_i d\mathcal{L}^n$ can be bounded as before. The factorization of the integrand

$$\sqrt{\sum_{|\alpha|+|\beta|=n} E^{|\beta|}((H_t)_\#(\Psi_\#(P))_{\alpha\beta})^2} = \sqrt{\prod_j \left(1 + E\left\|P_j\right\|^2\right)},$$

since we only are concerned with points at non-vertical tangents, $P_j = \nabla_j(F + t\eta)$, is well-defined, where the covariant derivative is in the direction of $\partial/\partial X_j$ as before, and the basis is chosen to diagonalize the quadratic form $(V, W) \mapsto < \nabla_V F + t\eta, \nabla_W F + t\eta >$ as in the previous case, A is this quadratic form at $t = 0$, and $\mathcal{A}$ is derived from A as before. Each such Q_i can also be bounded as (since $E < 1$) by

$$\begin{aligned} |(\langle \nabla_i \eta, \nabla_i F + t\nabla_i \eta \rangle) Q_i| &\ll \sqrt{E}\sqrt{\Pi_{j\neq i}\left(1 + E\left\|P_j\right\|^2\right)} \\ &\ll \sqrt{E}\sqrt{\prod_j\left(1 + E\left\|P_j\right\|^2\right)} \\ &= \sqrt{E}\sqrt{\sum_{|\alpha|+|\beta|=n} E^{|\beta|}((H_t)_\#(\Psi_\#(P))_{\alpha\beta})^2}, \end{aligned}$$

so that

$$\begin{aligned} &\left|\frac{d}{dt}K(t) - \int_{D_\epsilon}\sum_{i,k}\mathcal{A}_{ik}\left\langle \nabla_i\eta, \nabla_k F + t\nabla_k \eta\right\rangle d\mathcal{L}^n\right| \\ &\ll \frac{1}{ER^n}\int_P \frac{d}{dt}\sqrt{\sum_{|\alpha|+|\beta|=n} E^{|\beta|}((H_t)_\#(\Psi_\#(P))_{\alpha\beta})^2} d\left\|P\right\| \\ &\ll \left\|T_\epsilon \llcorner \pi^{-1}(D_\epsilon)\right\| / (\sqrt{E}R^n) \\ &\ll \sqrt{E} \end{aligned}$$

by Corollary (7.4). This establishes the Lemma. □

LEMMA 9.2. *With the hypotheses of Lemma (9.1), if the support of η is contained in $|X| < 1 - E^{1/4n}$ and $|\nabla\eta| \leq 1$, we also have*

$$\left|\int_{B(X_0,1)} \sum \mathcal{A}_{ik}\left\langle \nabla_i \eta, \nabla_k F\right\rangle d\mathcal{L}^n\right| \ll \sqrt{E}.$$

Proof. Here we use the minimality of T_ϵ. From Lemma (9.1), we have that

$$\left|\frac{d}{dt}\mathcal{F}_{\epsilon,R}\left((h_t)_\#\left(T_{\epsilon,R}\right)\right)/(ER^n) - \int_{B(X_0,1)}\sum_{i,k}\mathcal{A}_{ik}\left\langle \nabla_i\eta, \nabla_k F + t\nabla_k\eta\right\rangle d\mathcal{L}^n\right| \ll \sqrt{E}.$$

However, since T_ϵ minimizes $\mathcal{F}_\epsilon$, $T_{\epsilon,R}$will minimize $\mathcal{F}_{\epsilon,R}$ by its definition. This implies that

$$\left.\frac{d}{dt}\right|_0 \mathcal{F}_{\epsilon,R}\left((h_t)_\#\left(T_{\epsilon,R}\right)\right) = 0,$$

and the Lemma follows from setting $t = 0$. □

LEMMA 9.3. *For any* $L : B(x_0, R) \to \mathbb{R}^k$ *so that, for some* σ, $|grad(L)| \le \sigma \le 1.$, *let* $h(x,y) = (x, y - L(x))$. *Then*

$$Exc(h_\#(T), R) \ll E + \sigma^2.$$

Proof. Since h is vertical, if $\mathbf{e} = dx^1 \wedge \cdots \wedge dx^n$ is the horizontal n-vector in $\Lambda_n(B(x_0,R) \times \mathbb{R}^k)$,

$$< \mathbf{e}, h_\# \left(\overrightarrow{T_\epsilon}\right) >=< \mathbf{e}, \overrightarrow{T_\epsilon} >$$

and so, for any multiindex

$$\left|< dx^\alpha \wedge dy^\beta, h_\# \left(\overrightarrow{T_\epsilon}\right) > - < dx^\alpha \wedge dy^\beta, \overrightarrow{T_\epsilon} >\right| \ll \sigma.$$

Since

$$\begin{aligned}
\left\|h_\#(\overrightarrow{T})\right\| &= \sqrt{\sum_{|\alpha|+|\beta|=n} < dx^\alpha \wedge dy^\beta, h_\#(\overrightarrow{T}) >^2} \\
&\le \sqrt{< \mathbf{e}, \overrightarrow{T} > + \sum_{|\alpha|+|\beta|=n, |\beta|>0} (< dx^\alpha \wedge dy^\beta, \overrightarrow{T} > + c\sigma)^2} \\
&\le \sqrt{1 + c'\sigma \left(\sum_{|\alpha|+|\beta|=n, |\beta|>0} < dx^\alpha \wedge dy^\beta, \overrightarrow{T} > \right) + c'\sigma^2} \\
&\le \sqrt{1 + c''\sigma \sqrt{\sum_{|\alpha|+|\beta|=n, |\beta|>0} (< dx^\alpha \wedge dy^\beta, \overrightarrow{T} >)^2} + c''\sigma^2} \\
&\le 1 + c''\sigma \sqrt{1 - < \mathbf{e}, \overrightarrow{T} >^2} + c''\sigma^2,
\end{aligned}$$

$$\begin{aligned}
\|h_\#(T)\| &\le (1 + c''\sigma^2)\|T\| + c''\sigma \int_{C(x_0,R)} \sqrt{1 - < \mathbf{e}, \overrightarrow{T} >^2} d\|T\| \\
&\le (1 + c''\sigma^2)\|T\| + c'' \left(\sigma \sqrt{\|T\|}\right) \sqrt{ER^n} \\
&\le \|T\| + c'''\sigma^2 \|T\| + c''' ER^n.
\end{aligned}$$

Since $\|T\| \ll R^n$, the Lemma follows. □

10. Iterative inequality

Fix β, $0 < \beta \le 1/4$.

PROPOSITION 10.1. *If* T *is a mass-minimizing rectifiable section* $T \in \widetilde{\Gamma}(B)$ *which is the limit of a sequence of penalty minimizers* T_ϵ, *and there exists a positive constant* $\alpha = \alpha(\beta)$ *and a constant* c, *so that if*

$$R + Exc(T; R) \le \alpha,$$

then

$$Exc(h_\# T; \beta R) \le c\beta^2 Exc(T; R) \tag{10}$$

for some linear map $h(x,y) = (x, y - l(x))$ *with*

$$|grad\ l| \le \alpha^{-1} \sqrt{Exc(T; R)}. \tag{11}$$

Remark 10.2. Note that, if this Lemma holds with some one value of α, it will also hold with any smaller α. Also, recall from Theorem [3.1] that for any homology class of rectifiable sections there will be one such section which is the limit of penalty minimizers.

Proof. If this is not the case, then we will be able to find a sequence $R_i \to 0$, $\epsilon_i \to 0$, along with functionals $\mathcal{F}_i := \mathcal{F}_{\epsilon_i,R_i}$ as above and $T_i \to T$ (minimizers of $\mathcal{F}_i$), and excesses $E_i := Exc(T_{\epsilon_i}; R_i, x_0)$ for which $E_i \to 0$ and (by choosing each R_i sufficiently small) $E_i^{1/4n}/\epsilon_i \to 0$, and

$$\limsup_{i\to\infty} E_i^{-1} Exc((h_i)_\#(T_i); \beta R_i) \geq c\beta^2 \tag{12}$$

for all linear maps $h_i(x,y) = (x, y - l_i(x))$with

$$\limsup_{i\to\infty} E_i^{-1/2} |\text{grad } l_i| < \infty.$$

Such a sequence $\{T_i,\, \mathcal{F}_i,\, R_i\}$, following [2], will be called an *admissible sequence.*

As before, let D_{ϵ_i} be the bad set over which $T_i := T_{\epsilon_i}$ is not necessarily a C^1 graph with bounded gradient, and let $D_i := \phi_{R_i}(D_{\epsilon_i}) \cap B(X_0, 1)$. Then, on $B(X_0,1)\backslash D_i$, $T_i := T_{\epsilon_i,R_i}$will be the graph of a C^1 function G_i, agreeing on $B(X_0,1)\backslash D_i$ with F_i, which is the BV carrier of T_i on $B(X_0, R)$. We need to show:

LEMMA 10.3. *For all i sufficiently large*

1.
$$\int_{B(X_0,1)} \|dF_i\|\, d\mathcal{L}^n \ll 1,$$

2.
$$\lim_i \|D_i\| = 0$$

3.
$$\lim_i \frac{\int_{B(X_0,1/2)\backslash \phi_{R_i}(D_i)} |F_i|^{2p}\, d\mathcal{L}^n}{(E_i)^{p/2n}} \ll_p 1,\ \ 1 \leq p < \frac{n}{n-1},$$

4.
$$\int_{B(X_0,1)} |F_i|\, d\mathcal{L}^n \ll 1$$

5.
$$\frac{Exc(T_i, R_i/4)}{E_i} \ll \left(2 + \frac{1}{2\epsilon_i}\right) E_i^{1/4n} + \left(2 + \frac{1}{2\epsilon_i}\right)^{3/2} E_i^{3/4n},$$

6. *The limit*
$$\lim_i \mathcal{A}_i := \mathcal{A}_0$$
is the symbol of an elliptic PDE.

7. *for every smooth $\eta(X)$ with compact support in $|X| < 1$ we have*

$$\lim_i \int_{B(x_0,1)} \sum (A_i)_{jk} \left\langle \frac{\partial \eta}{\partial X_j}, D_k F_i \right\rangle d\mathcal{L}^n = 0,$$

8. *Finally, if $h_i(x,y) = (x, y - l_i(x))$ is a sequence of linear maps with*

$$\lim_i \frac{|grad(l_i)|}{\sqrt{E_i}} \leq \sigma$$

then

$$\lim_i \frac{Exc((h_i)_\#(T_i), R_i)}{E_i} \ll (1 + \sigma^2).$$

Proof. Set, for each i in the sequence,

$$\overline{y}(i) := \frac{1}{\|B(x_0, R/2) \backslash (D_\epsilon \cap B(x_0, R/2))\|} \int_{B(x_0,R/2)\backslash(D_\epsilon \cap B(x_0,R/2))} f_i d\mathcal{L}^n.$$

For each i, translate the corresponding graph so that so that $\overline{y}(i) = \overline{0}$. By Lemma(8.7), there is a constant τ so that for all p, $1 \leq p \leq \frac{n}{n-1}$,

$$\left(\frac{\int_{B(x_0,R)} |f_i|^p \, d\mathcal{L}^n}{R^n} \right)^{1/p} \leq \tau \frac{\int_{B(x_0,R)} \|df_i\| \, d\mathcal{L}^n}{R^{n-1}} := \tau \frac{\int_{B(x_0,R)} \|df_i\|}{R^{n-1}},$$

and since we have by Lemma (8.2) that $\int_{B(x_0,R)} \|df_i\| \ll \sqrt{E_i} R^n$, with $p = 1$we conclude that

$$\int_{B(x_0,R)} |f_i| \, d\mathcal{L}^n \ll \sqrt{E_i} R^{n+1},$$

which since $F_i(X) = f_i(RX)/(\sqrt{E_i}R)$, as before yields that for all i sufficiently large,

$$\int_{B(X_0,1)} \|dF_i\| \, d\mathcal{L}^n \ll 1, \text{ and } \int_{B(X_0,1)} |F_i| \, d\mathcal{L}^n \ll 1,$$

which are statements (4) and (1), respectively. Lemma (8.8) and the definition of F_i immediately gives statement (3), and statement (2) follows from the bound $\|D_\epsilon\| \leq ER^n$, so that $\|D_i\| \ll E_i$, and the choice of $E_i \downarrow 0$.

To show statement (5), use Lemma(8.10) to show that (with $\overline{y} = 0$) and Lemma (8.8)

$$\begin{aligned}
\frac{Exc(T_i, R_i/4)}{E_i} &\ll (1 + \frac{1}{2\epsilon_i}) \left(\mu + E_i^{1/2n} \left(\frac{1}{\mu} \right) \right) + \int_{B(x_0,R_i/2)\backslash D_{\epsilon_i}} |f_i|^2 \, d\mathcal{L}^n \Big/ (\mu E_i R_i^{n+2}) \\
&\ll (1 + \frac{1}{2\epsilon_i}) \left(\mu + E_i^{1/2n} \left(\frac{1}{\mu} \right) \right) + \int_{B(x_0 1/2)} |F_i|^2 \, d\mathcal{L}^n \Big/ (\mu) \\
&\ll (1 + \frac{1}{2\epsilon_i}) \left(\mu + E_i^{1/2n} \left(\frac{1}{\mu} \right) \right) + \int_{B(x_0 1/2)} |F_i|^2 \, d\mathcal{L}^n \Big/ (\mu) \\
&= \mu(1 + \frac{1}{2\epsilon_i}) + \frac{1}{\mu} \left((1 + \frac{1}{2\epsilon_i}) E_i^{1/2n} + \int_{B(x_0 1/2)} |F_i|^2 \, d\mathcal{L}^n \right).
\end{aligned}$$

Taking μ to minimize the right hand side above,

$$\mu = \mu_i = \frac{\sqrt{(1+\frac{1}{2\epsilon_i})E_i^{1/2n} + \int_{B(x_0 1/2)} |F_i|^2 \, d\mathcal{L}^n}}{\sqrt{1+\frac{1}{2\epsilon_i}}},$$

which for i sufficiently large will be less than one, by (3) above, and the fact that $E_i \searrow 0$, gives

$$\begin{aligned}\frac{Exc(T_i, R_i/4)}{E_i} &\ll \sqrt{1+\frac{1}{2\epsilon_i}}\sqrt{(1+\frac{1}{2\epsilon_i})E_i^{1/2n} + \int_{B(x_0 1/2)} |F_i|^2 \, d\mathcal{L}^n} \\ &\quad + \left((1+\frac{1}{2\epsilon_i})E_i^{1/2n} + \int_{B(x_0 1/2)} |F_i|^2 \, d\mathcal{L}^n\right)^{3/2} \\ &\ll \sqrt{1+\frac{1}{2\epsilon_i}}\sqrt{2+\frac{1}{2\epsilon_i}}E^{1/4n} + \left(2+\frac{1}{2\epsilon}\right)^{3/2} E^{3/4n}.,\end{aligned}$$

easily giving (5).

Statement (6) follows from Equation (5). Statement (7) follows from Lemma (9.2).

Finally, statement (8) follows from Lemma (9.3). □

By statements (1) and (3) of this Lemma, invoking the closure and compactness theorems for BV functions [5], we can assume that there is an element $F \in BV(B(X_0, 1))$ so that a subsequence (which by standard abuse of notation we do not re-label) $F_i \to F$ strongly in $L^1(B(X_0, 1))$ and $DF_i \to DF$ as distributions. We then have

$$\int_{B(X_0,1)} \sum A_{jk} \left\langle \frac{\partial \eta}{\partial X_j}, D_k F \right\rangle d\mathcal{L}^n = 0,$$

for all smooth η with compact support in $|X| < 1$. Thus, F will be $\mathcal{A}$-harmonic, and thus is a real-analytic function. It then follows from the Di Giorgi-Moser-Morrey estimates for diagonal elliptic systems [12] that

$$\sup_{B(X_0,1/2)} |F| \ll \int_{B(X_0,1)} |F| \, d\mathcal{L}^n = \lim_i \int_{B(X_0,1)} |F_i| \, d\mathcal{L}^n \ll 1,$$

so we can shift the graph so that $F(X_0) = 0$. Our previous shift was chosen so that, for each i, $\overline{y}(i) = 0$, where

$$\overline{y}(i) := \frac{1}{\|B(x_0, R_i/2) \setminus (D_{\epsilon_i} \cap B(x_0, R_i/2))\|} \int_{B(x_0,R_i/2)\setminus \left(D_{\epsilon_i} \cap B(x_0,R_i/2)\right)} f_i d\mathcal{L}^n.$$

The bounds on the L^1 norms of F and the BV-norm of DF are not worsened by this assumption except for possible change of constants, which are implicit in the notation. In addition, the bound of statement (3) in Lemma(10.3) continues to hold as well, since the bound $\overline{y}(i) = 0$ becomes

$$\int_{B(X_0,1/2)\setminus \phi_{R_i}(D_i)} F_i d\mathcal{L}^n = 0,$$

from which follows the fact that

$$\int_{B(X_0,1/2)\setminus \phi_{R_i}(D_i)} |F_i|^2 \, d\mathcal{L}^n \leq \int_{B(X_0,1/2)\setminus \phi_{R_i}(D_i)} |F_i + C|^2 \, d\mathcal{L}^n$$

for any constant vector C.

LEMMA 10.4. *Let $\{T_i, \mathcal{F}_i, R_i\}$ be admissible. Under a suitable translation (or change of coordinates), $F(0) = 0$, $F_i \to F$ strongly in L^1, F is a solution to the equation*

$$\int_{B(X_0,1)} \sum (\mathcal{A})_{jk} \left\langle \frac{\partial \eta}{\partial X_j}, D_k F \right\rangle d\mathcal{L}^n = 0,$$

as well as

$$\int_{B(X_0,1)} |F| \, d\mathcal{L}^n + \int_{B(X_0,1)} |gradF| \, d\mathcal{L}^n \ll 1,$$

$$\sup_{B(X_0,1/2)} (|F|, |gradF|) \ll 1,$$

and

$$\lim_i \frac{Exc(T_i, R_i/4)}{E_i} = 0$$

under the assumption that $\lim_i E_i^{1/4n}/\epsilon_i = 0$.

Proof.

$$\int_{B(X_0,1)} |F| \, d\mathcal{L}^n = \lim_i \int_{B(X_0,1)} |F_i| \, d\mathcal{L}^n$$

because $F_i \to F$ strongly in L^1, and

$$\int \|DF\| \, d\mathcal{L}^n \leq \lim_i \int_{B(X_0,1)} \|DF_i\| \, d\mathcal{L}^n$$

by lower semi-continuity with respect to BV-convergence. In order to complete the proof of the Lemma, we need only show that

$$\lim_i \int_{B(X_0,1/2) \setminus \phi_{R_i}(D_i)} |F_i|^2 \, d\mathcal{L}^n = \int_{B(X_0,1/2)} |F|^2 \, d\mathcal{L}^n.$$

But, if Φ_i is the characteristic function of $B(X_0, 1) \setminus \phi_{R_i}(D_i)$, then

$$\begin{aligned}
\lim_i \Bigg| \int_{B(X_0,1/2) \setminus \phi_{R_i}(D_i)} & |F_i|^2 \, d\mathcal{L}^n \\
- \int_{B(X_0,1/2)} |F|^2 \, d\mathcal{L}^n \Bigg| &= \lim_i \left| \int_{B(X_0,1/2)} \Phi_i |F_i|^2 - |F|^2 \, d\mathcal{L}^n \right| \\
&= \lim_i \left| \int_{B(X_0,1/2)} \Phi_i \left(|F_i|^2 - |F|^2 \right) + \Phi_i |F|^2 - |F|^2 \, d\mathcal{L}^n \right| \\
&= \lim_i \left| \int_{B(X_0,1/2)} \Phi_i (|F_i| - |F|)(|F_i| + |F|) + (\Phi_i - 1) |F|^2 \, d\mathcal{L}^n \right| \\
&\leq \lim_i \left| \int_{B(X_0,1/2)} \Phi_i |F_i - F| (|F_i| + |F_i|) + (\Phi_i - 1) |F|^2 \, d\mathcal{L}^n \right| \\
&\leq \lim_i \int_{B(X_0,1/2)} \Phi_i (|F_i| + |F|) |F_i - F| \, d\mathcal{L} \\
&\quad + \int_{B(X_0,1/2)} (1 - \Phi_i) |F|^2 \, d\mathcal{L}^n.
\end{aligned}$$

Since F is uniformly bounded in $B(X_0, 1/2)$ and $F_i \to F$ strongly in L^1, a subsequence will converge almost-everywhere pointwise, and $\lim_i \int_{B(X_0,1)}(1-\Phi_i)d\mathcal{L}^n = 0$, the last integral above goes to 0, and

$$\begin{aligned}\lim_i \left| \int_{B(X_0,1/2)} \Phi_i \, |F_i|^2 - |F|^2 \, d\mathcal{L}^n \right| &\le 2 \lim_i \int_{B(X_0,1/2)} \Phi_i \, |F_i| \, |F_i - F| \, d\mathcal{L} \\ &\le \lim_i \left(\int_{B(X_0,1/2)} \Phi_i \, |F_i|^{2p-1} \, |F_i - F| \, d\mathcal{L} \right)^{\frac{1}{2p-1}} \cdot \\ &\quad \cdot \left(\int_{B(X_0,1/2)} |F_i - F| \, d\mathcal{L} \right)^{1-\frac{1}{2p-1}}.\end{aligned}$$

The last step is Hölder's inequality for the measure $\mu = |F_i - F| d\mathcal{L}^n$. If $1 < p < \frac{n}{n-1}$ then the first of these last two integrals is uniformly bounded by statement (3) of Lemma (10.3) by a power of E_i (Note that $E_i \to 0$ as $i \to \infty$.), and the height bound on F_i and F coming from the compactness of the fiber of the bundle. The second integral goes to 0 in the limit by the strong convergence of F_i to F in L^1. □

We can now complete the proof of Proposition (10.1). Let $L = L(X)$ denote the linear forms

$$L(X) := \sum \frac{\partial F}{\partial X_i}(0) X_i,$$

and let h_i be the maps

$$h_i(x, y) = \left(x, y - \sqrt{E_i} L(x) \right),$$

and

$$\widetilde{T}_i := (h_i)_\#(T_i).$$

Set $\widetilde{E}_i := Exc(\widetilde{T}_i, R_i)$. Since $|gradL| = |DF| \ll 1$, we apply statement (8) of Lemma (10.3), which shows that

$$\lim_i \frac{\widetilde{E}_i}{E_i} \ll 1.$$

Case 1. $\lim_i \widetilde{E}_i / E_i = 0$. This contradicts

$$\limsup_{i \to \infty} E_i^{-1} Exc((h_i)_\#(T_i); \beta R_i) \ge c\beta^2,$$

which is a basic assumption on the sequence T_i, since this case implies that $\lim_i Exc(\widetilde{T}_i, \beta R_i)/E_i \le \lim_i \widetilde{E}_i / (\beta^n E_i) = 0$.

Case 2. $\lim_i \widetilde{E}_i / E_i > 0$. Then, the currents $\widetilde{T}_i$ minimize $\mathcal{F}_i$, so that $\{\widetilde{T}_i, \mathcal{F}_i, R_i\}$ will still be admissible, so that

$$\widetilde{F}_i \to \widetilde{F},$$

where

$$\widetilde{F}_i(X) := \sqrt{\widetilde{E}_i^{-1} E_i} \, (F_i - L)(X)., \; \widetilde{F}(X) := \lim_i \sqrt{\widetilde{E}_i^{-1} E_i} \, (F - L)(X).$$

Since $\widetilde{F}$ satisfies the conditions of Lemma (10.4), in particular

$$\lim_i \widetilde{E_i}^{-1} Exc(\widetilde{T}_i, R_i/4) = 0$$

and in addition we have

$$D\widetilde{F}(0) = 0,$$

and the inequality (12) in the beginning of the proof Proposition (10.1), becomes

$$\lim_i E_i^{-1} Exc(\widetilde{T}_i, \beta R_i) \geq c\beta^2.$$

Define s as the integer so that

$$\frac{1}{4} \leq 4^s \beta < 1$$

(assume that $\beta < 1/4$, so that $s \geq 1$), and for $\sigma = 0, 1, 2, \ldots, s$ we consider

$$\widetilde{E_i}^{(\sigma)} := Exc(\widetilde{T}_i, 4^\sigma \beta R_i).$$

It is clear by the fact that $\lim_i \frac{\widetilde{E_i}}{E_i} \ll 1$ that

$$\widetilde{E_i}^{(\sigma)} \leq (4^\sigma \beta)^{-n} \widetilde{E_i} \ll (4^\sigma \beta)^{-n} E_i,$$

that is, for some constant C,

$$\widetilde{E_i}^{(\sigma)} \leq C(4^\sigma \beta)^{-n} E_i.$$

If, for some σ we have

$$\lim_i E_i^{-1} \widetilde{E_i}^{(\sigma)} = 0,$$

then we have

$$\lim_i E_i^{-1} Exc(\widetilde{T}_i, \beta R_i) = 0,$$

contradicting our assumption above. So, we can assume that

$$\lim_i E_i^{-1} \widetilde{E_i}^{(\sigma)} > 0.$$

We also have

$$\lim_i E_i^{-1} \widetilde{E_i}^{(\sigma)} \ll (4^\sigma \beta)^{-n} < +\infty,$$

for $\sigma = 0, 1, \ldots, s$. Now, these inequalities and the fact that $\{\widetilde{T}_i, \mathcal{F}_i, R_i\}$ is admissible implies that also

$$\{\widetilde{T}_i \llcorner C(x_0, 4^\sigma \beta R_i), \mathcal{F}_i \circ h_{\#}^{-1}, 4^\sigma \beta R_i\}$$

will be admissible. Thus, by the conclusions of Lemma (10.4),

$$\lim_i \frac{\widetilde{E_i}^{(\sigma-1)}}{\widetilde{E_i}^{(\sigma)}} = 0,$$

or, given any $a > 0$, for i sufficiently large,

$$\frac{\widetilde{E_i}^{(\sigma-1)}}{\widetilde{E_i}^{(\sigma)}} < a.$$

Iterating this inequality,

$$Exc(\widetilde{T}_i, \beta R_i) := \widetilde{E}_i^{(0)} < a\widetilde{E}_i^{(1)} < \cdots < a^\sigma \widetilde{E}_i^{(\sigma)} \leq Ca^\sigma (4^\sigma \beta)^{-n} E_i.$$

Choosing a sufficiently small will then guarantee that, for i sufficiently large

$$\frac{Exc(\widetilde{T}_i, \beta R_i)}{E_i} < c\beta^2,$$

contradicting the assumption, and completing the proof of Proposition (10.1).

□

There is a small extension of this Proposition that will be needed for its application:

COROLLARY 10.5. *Given β, T, $\alpha(\beta)$ as in Proposition (10.1), then the conclusion of the Proposition will still hold, for some $\alpha > 0$, for the current $H_\#(T)$, where $H(x,y) = (x, y + L(x))$ is a fixed linear map. That is, if*

$$R + Exc(H_\#(T), R) \leq \alpha, \tag{13}$$

then

$$Exc(h_\#(H_\#(T)), \beta R) \leq c\beta^2 Exc(H_\#(T), R) \tag{14}$$

for some linear map $h(x,y) = (x, y - l(x))$ with

$$|grad\ l| \leq \alpha^{-1}\sqrt{Exc(H_\#(T), R)}. \tag{15}$$

Proof. This corollary will follow from Proposition (10.1) once it is shown that, under these conditions, $|\text{grad } l|$ is bounded as indicated in (15). However, under the assumptions on R and the excess of $H_\#(T)$, if f is the BV-carrier of T, so that $(f+L)$ is the BV-carrier of $H_\#(T)$,

$$\begin{aligned}\int_{B(0,R)} \|D(f+L)\|\, d\mathcal{L}^n &\geq \int |\|Df\| - \|\text{grad } L\||\, d\mathcal{L}^n \\ &\geq \int_{B(0,R)} \|\text{grad } L\|\, d\mathcal{L}^n - \int \|Df\|\, d\mathcal{L}^n\end{aligned}$$

so, by Lemma (8.2)

$$\begin{aligned}\|\text{grad } L\|\, R^n &\leq \int_{B(0,R)} \|Df\|\, d\mathcal{L}^n + \int_{B(0,R)} \|D(f+L)\|\, d\mathcal{L}^n \\ &\leq \sqrt{E}R^n + \|H_\#(T)\| \\ &\leq \sqrt{E}R^n + (\alpha+1)R^n.\end{aligned}$$

Thus, by Proposition (10.1), which implies that there is a linear map h for which $Exc(h_\#(T), \beta R) \leq c\beta^2 Exc(T, R)$, so that, when k is given by $k := l - L$, k satisfies the excess conditions (13) and (14) of this Corollary and

$$\|\text{grad } k\| \leq \|\text{grad } l\| + \|\text{grad } L\|,$$

which, for $\alpha > 0$ sufficiently small satisfies the gradient bound condition (15). □

The primary use of Proposition (10.1) and its corollary is in the following Lemma. Let $Exc(T, a, r)$ be the excess of T over $B(a, r)$.

LEMMA 10.6. *There is a positive constant E_0 with the following property. If T is as in Proposition (10.1) and*

$$R + Exc(T, 0, R) \leq E_0,$$

then, for all a, r with $|a| < R/2$, $r \leq R/2$ there is a linear map $h(x, y) = (x, y - l(x))$ so that

$$Exc(h_{\#}(T), a, r) \leq C \left(\frac{r}{R}\right)^2 Exc(T, 0, R).$$

Moreover, $|grad(l)| \leq 1/E_0$.

Proof. Since

$$Exc(T, a, R/2) \leq 2^n Exc(T, 0, R),$$

by replacing $E_0$by $E_0/2^n$ we see that it is sufficient to prove the Lemma in the special case $a = 0$. To do so, we apply Proposition (10.1) and Corollary (10.5) several times. Each time, we may need to use a smaller α, but since our iteration is finite there will be a sufficiently small α to work for all steps simultaneously. We get linear maps h_1, h_2, $\ldots$, h_s, $h_i(x, y) = (x, y - l_i(x))$ so that

$$Exc((h_i)_{\#}(T), 0, \beta^i R) \leq c\beta^2 Exc((h_{i-1})_{\#}(T), 0, \beta^{i-1} R)$$

for $i = 1, \ldots, s$, and

$$|grad(l_i - l_{i-1})| \leq \alpha^{-1} \sqrt{Exc((h_{i-1})_{\#}(T), 0, \beta^{i-1} R)},$$

with $l_0 = 0$, and α satisfying the conditions of Proposition (10.1) or Corollary (10.5) for $(h_i)_{\#}(T)$. Iterating the first inequality s times,

$$\begin{aligned}
Exc((h_s)_{\#}(T), 0, \beta^s R) &\leq c\beta^2 Exc((h_{s-1})_{\#}(T), 0, \beta^{s-1} R) \\
&\leq c^2 \beta^4 Exc((h_{s-2})_{\#}(T), 0, \beta^{s-2} R) \\
&\ \ \vdots \\
&\leq c^s \beta^{2s} Exc(T, 0, R).
\end{aligned}$$

So, choosing s so that $\beta^{s+1} R < r \leq \beta^s R$, and $h = h_s$ we have the second claim of the Lemma. The second inequality from Proposition (10.1) or its corollary becomes

$$|grad(l_i - l_{i-1})| \leq \alpha^{-1} c^{(i-1)/2} \beta^{i-1} \sqrt{Exc(T, 0, R)},$$

so

$$\begin{aligned}
|grad(l_s)| &\leq \\
\sum_{i=1}^{s} |grad(l_i - l_{i-1})| &\leq \sum_{i=1}^{s} \alpha^{-1} c^{(i-1)/2} \beta^{i-1} \sqrt{Exc(T, 0, R)},
\end{aligned}$$

which, assuming at no loss in generality that $c\beta\sqrt{E} < 1/2$, is less than $2/\alpha$. Choosing $E_0 = \alpha/2$ completes the proof. $\square$

PROPOSITION 10.7. *With the hypotheses of Lemma (10.6), the BV-carrier function $f(x)$ of T is of class C^1 in $B(0, R/2)$.*

Proof. Recall that $T \llcorner y_i = B(0,R) \llcorner f_i$. If $h(x,y) = (x, y - l(x))$,with l linear, then the corresponding function for $h_\#(T)$ is of course $f - l$. Apply Lemma (8.2) to the current $h_\#(T) \llcorner C(a,r)$, implying from Lemma (10.6) that

$$\int_{B(a,r)} |Df - l| \, d\mathcal{L}^n \ll r^n \sqrt{C} \left(\frac{r}{R}\right) \sqrt{E},$$

for all $a \in B(0, R/2)$ and all $r \le R/2$, for $l = l_{r,a} = D(l_{r,a})$, where, from Lemma (10.6), note that the linear map l of that lemma, here denoted $l_{r,a}$, depends on the center a and radius r of the ball. We need to show that the limit

$$l_a = \lim_{r \to 0} l_{r,a}$$

exists for all $a \in B(0, R/2)$. Since

$$\begin{aligned} \int_{B(a,r/2)} \left| l_{r,a} - l_{r/2,a} \right| d\mathcal{L}^n \;&\le\; \int_{B(a,r/2)} \left| Df - l_{r/2,a} \right| d\mathcal{L}^n + \int_{B(a,r)} |Df - l_{r,a}| \, d\mathcal{L}^n \\ &\ll\; r^n \sqrt{C} \left(\frac{r}{R}\right) \sqrt{E}, \end{aligned}$$

so, by the fact that the first integrand above is constant,

$$\left| l_{r,a} - l_{r/2,a} \right| \ll \sqrt{C} \left(\frac{r}{R}\right) \sqrt{E}.$$

Iterating that inequality and adding,

$$\left| l_{r,a} - l_{r/2^n,a} \right| \ll \sqrt{C} \left(\frac{r}{R}\right) \sqrt{E} \sum_{j=0}^{\infty} 2^{-j},$$

so, by the triangle inequality, $\left\{ l_{r/2^n,a} \right\}$ is Cauchy in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^j)$. Set $\hat{l}_a := \lim l_{r/2^n,a}$, then $\left| l_{r,a} - \hat{l}_a \right| \ll r/R$, and so

$$l_a := \lim_{r \to 0} l_{r,a} = \hat{l}_a$$

exists.

By a similar argument to the above, for $a, b \in B(0, R/2)$, with $|a - b| < r/2$,

$$|l_{r,a} - l_{r,b}| \ll r/R,$$

and so

$$|l_a - l_b| \ll r/R$$

if $|a - b| < r/2$, and so $a \mapsto l_a$ is continuous in $B(0, R/2)$.

From this it follows that $r^{-n} \int_{B(a,r)} |Df| \, d\mathcal{L}^n$ is uniformly bounded for $a \in B(0, R/2)$, $r < R/2$, and so the measure $|Df| \, d\mathcal{L}^n$ is absolutely continuous, so that we can write

$$Df \, d\mathcal{L}^n = \phi d\mathcal{L}^n$$

for some $\phi \in L^1(B(0, R/2))$. Since $\phi \in L^1$, almost every point in $B(0, R/2)$ is a Lebesgue point, and

$$\lim_{r \to 0} r^{-n} \int_{B(a,r)} |\phi(x) - \phi(a)| \, d\mathcal{L}^n = 0.$$

But we already have that

$$\lim_{r \to 0} r^{-n} \int_{B(a,r)} |\phi(x) - l_a| \, d\mathcal{L}^n = 0,$$

so that $\phi(a) = l_a$ almost-everywhere in $B(0, R/2)$, so $Df = \phi d\mathcal{L}^n$ with now ϕ continuous on $B(0, R/2)$, and so $f \in C^1(B(0, R/2))$. $\square$

Similarly to the discussion in [2, p. 129, lines 12-24], we have:

Let $z \in Supp(T)$ be a point with an approximate tangent plane $Tan(Supp(T), z)$. By the rectifiability theorem for currents and our lower bound on density Proposition (4.1), we have that $Tan^n(\|T\|, z) = Tan(Supp(T), z)$, and is an n-dimensional vector space. If the oriented tangent plane $\overrightarrow{T_z}$ is not vertical, that is, $\pi_* \overrightarrow{T_z} = \mathbf{e}$, then there is a linear map $H(x, y) := (x, y - L(x))$ for which $\overrightarrow{H_\#(T)_{H(z)}} = \mathbf{e}_0$. By Corollary (10.5), Lemma (10.6) and so Proposition (10.7) will apply to $H_\#(T)$ as well. By the monotonicity result, the density $\Theta(T, z) = 1$ at each point. As a final assumption, assume that the tangent cone $Tan(Supp(T), z) \subset V$, where $V \subset \mathbb{R}^{n+j}$ is an n-dimensional plane. Since $Tan(Supp(T), Z) \supset Supp(\overrightarrow{T_z})$, the plane $V = Supp(\overrightarrow{T_z})$ is not vertical. Apply a shear-type linear map $H(x, y) := (x, y + L(x))$ so that $H(Tan(Supp(T), Z) = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+j}$. Presume that coordinates are chosen so that $z = (0, 0)$.

PROPOSITION 10.8. *Under the conditions of Lemma (10.7), $Supp(T)$ is a C^1, n-dimensional graph over some ball $B(0, r)$.*

Proof. It of course suffices to show that $Supp(H_\#(T))$ is a C^1 graph over $B(0, r)$. Since the tangent plane of $H_\#(T)$ over 0 is the horizontal plane in the coordinate system of the last line of the previous paragraph, given $\eta > 0$, there is an $r = r_\eta > 0$ so that

$$Supp(H_\#(T) \llcorner C(0, r)) \subset \{|x| \leq r, |y| \leq \eta r\} = B(0, r) \times B(0, \eta r), \text{ if } r \leq r_\eta,$$

and

$$Supp(\partial(H_\#(T) \llcorner B(0, r) \times B(0, \eta r))) \subset \partial B(0, r) \times B(0, \eta r).$$

Once we show that

$$\lim_{r \to 0} Exc(H_\#(T), 0, r) = 0,$$

then we can apply Lemma (10.6) and Proposition (10.7) to complete the proof of the present Proposition.

Let T be energy-minimizing among rectifiable sections, T_0 the current $B(0, r) \times \{0\} \subset B(0, r) \times B(0, \eta r)$, and F_r be the "fence" obtained by connecting each element of $(x, y) \in Supp(\partial H_\#(T) \llcorner B(0, r) \times B(0, \eta r))$ to $(x, 0) \in Supp(\partial T_0)$. Note that T_0 and $H_\#(T) \llcorner B(0, r) \times B(0, \eta r) - F_r$ have the same boundary, ∂T_0. It is easy to see ([5, p. 363], or [2, p. 128]) that,

$$\|F_r\| \leq \left(\sup_{\partial T} |y|\right) \left\|\partial H_\#(T) \llcorner B(0, r) \times B(0, \eta r)\right\|,$$

and, by slicing and the monotonicity formula, for a generic ρ, $r < \rho < 2r$ ($r < R/2$), there is a C so that $\left\|\partial H_\#(T) \llcorner B(0, \rho) \times B(0, \eta \rho)\right\| \leq C \rho^{n-1}$. Combining these two inequalities together,

$$\|F_\rho\| \leq C \eta \rho^n.$$

Since each penalty functional satisfies the ellipticity bounds (equation (1)),

$$\left[\left\|H_\#(T) \llcorner B(0, \rho) \times B(0, \eta r) - F_\rho\right\| - \|T_0\|\right] \leq \mathcal{F}_\epsilon(H_\#(T) \llcorner B(0, \rho) \times B(0, \eta \rho) - F_\rho) - \mathcal{F}_\epsilon(T_0),$$

then so will the limiting functional $\mathcal{F}$. Then, by subadditivity, and minimality of T,

$$\begin{aligned}\left[\left\|H_\#(T)\llcorner B(0,\rho)\times B(0,\eta\rho)-F_\rho\right\|-\|T_0\|\right] &\leq \mathcal{F}(H_\#(T)\llcorner B(0,\rho)\times B(0,\eta\rho)-F_\rho)-\mathcal{F}(T_0)\\ &\leq \|H\|^n\left[\mathcal{F}(T\llcorner H^{-1}(B(0,\rho)\times B(0,\eta\rho)))\right.\\ &\quad\left.-\mathcal{F}(H_\#^{-1}(T_0+F_\rho))+2\mathcal{F}(H_\#^{-1}(F_\rho))\right]\\ &\leq 2\|H\|^n\mathcal{F}(H_\#^{-1}(F_\rho))\\ &\leq 2C\eta\rho^n\end{aligned}$$

and so

$$\begin{aligned}Exc(H_\#(T),r) &= \left(\left\|H_\#(T)\llcorner B(0,r)\times B(0,\eta r)\right\|-\|T_0\|\right)\Big/ r^n\\ &\leq 2^n\left(\left\|H_\#(T)\llcorner B(0,\rho)\times B(0,\eta\rho)\right\|-\|T_0\|\right)\Big/\rho^n\\ &\leq 2^n\left[\|F_\rho\|+\left\|T\llcorner B(0,\rho)\times B(0,\eta\rho)-F_\rho\right\|-\|T_0\|\right]\Big/\rho^n\\ &\leq 2^n 3C\eta.\end{aligned}$$

Since, for any $\eta>0$ there is an $r>0$ sufficiently small so that the conditions of Proposition (10.7) hold, the conclusion of the Proposition holds. $\square$

This proposition shows that the set of "good" points in the base manifold M, the set of points where there is a non-vertical tangent space, is an open set, and on that open set the graph is of class C^1. The next result completes the proof of the main theorem, Theorem (0.1).

PROPOSITION 10.9. *Let T be an n-dimensional, mass-minimizing rectifiable section in $\widetilde{\Gamma}(B)$ which is the limit of a sequence of penalty-minimizers. Then, the projection $\pi(S)=Z$ onto* **e** *of the set S of all points $y\in Supp(T)$ so that the oriented tangent cone is not a plane, or where $T(T,y)$ has a vertical direction a closed set of Hausdorff n-dimensional measure 0 in M.*

Proof. The previous section shows that the set of points with non-vertical tangent planes is open in T, and projects to an open set. So, the set of points with no tangent plane, or with one having vertical directions, is closed. The set of points with no tangent plane is of measure 0 in any countably-rectifiable integer-multiplicity current, by [5, 3.2.19]. Assume there is a set $S_0\subset S$ of points of T with $Z_0:=\pi(S_0)$ of positive Hausdorff n-dimensional measure, and with vertical tangent planes. For all but a set of Hausdorff n-dimensional measure 0 in S_0, the density of the set will be 1 as well [5, 3.2.19]. For any such $z\in S_0$, given any $\epsilon>0$, there is a $\delta_{z,\epsilon}>0$ so that the ratio of the measure of of the projection onto M of $S_0\cap B(z,\delta_{z,\epsilon})$ to that of the ball of radius $\delta_{z,\epsilon}$centered at $\pi(z)$ in M will be less than ϵ,

$$\epsilon>\frac{\mathcal{H}^n(\pi(S_0\cap B_B(z,\delta_{z,\epsilon})))}{\mathcal{H}^n(B_M(\pi(z),\delta_{z,\epsilon}))}>\frac{\mathcal{H}^n(\pi(S_0\cap B_B(z,\delta_{z,\epsilon})))}{\omega_n\delta^n_{z,\epsilon}/2}$$

since the tangent plane is vertical. But the measure $\mathcal{H}^n(S_0\cap B_B(z,\delta_{z,\epsilon}))>\omega_n\delta^n_{z,\epsilon}/2$ for small enough $\delta_{z,\epsilon}$ since the density is one, so by the Besicovitch covering theorem

$$\mathcal{H}^n(S_0)>\frac{1}{\epsilon}\mathcal{H}^n(Z_0).$$

Since this must be true for any $\epsilon>0$, it would contradict the fact that T has finite mass if $\mathcal{H}^n(Z_0)>0$. $\square$

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